Removing the Mystery of Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$

by Tim Warriner. 5th January, 2019.
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[Version 2019-01-13-1618. Edits: Added section about calculating "i^{θ}" without knowledge of "$e^{i\theta}$". Replaced mentions of x as an angle with θ to make things clearer. Minor rephrasing. For the latest version, go to www.wimtarriner.com]

Introduction

The equations that get the most adoration in the maths world seem to be "$e^{i\theta} = \cos\theta + i\sin\theta$" and its variation, "$e^{i\pi} = -1 + 0i$", which is usually given as "$e^{i\pi} = -1$". In all the explanations that I have seen surrounding "$e^{i\theta}$", there is always a huge unexplained gap between the meaning of "e^{θ}" and the meaning of "$e^{i\theta}$". There is never any obvious reason given as to why an exponential curve should end up as a circle on the Complex plane, and generally, we just have to accept it all without question. It is hard to visualise why "$e^{i\theta}$" is what it is.

In this explanation, I will remove the obscurity of the meaning of "$e^{i\theta}$" and "$e^{i\theta} = \cos\theta + i\sin\theta$" by demonstrating exactly how "$e^{i\theta}$" works.

This entire explanation is based around the following ideas:

- It is possible to identify any point on a unit-radius circle by saying by how much the point at "1 + 0i" would have to be rotated to get there.

- It is possible to rotate a point on the Complex plane by multiplying it by powers of "i". I will use the symbol θ as the exponent in such powers. [At first, the use of this symbol might seem odd, but the exponent in this case is really an angle in a system where the circle has been divided into 4 pieces]. You can rotate a point on the Complex plane by multiplying it by "i^{θ}".

- Therefore you can identify a point on a unit-radius circle by saying by how much "1 + 0i" would have to be multiplied by "i^{θ}" to get there.

- Therefore, you can identify any point on a unit-radius circle solely in terms of "i^{θ}".

- Therefore, you can say that any point indicated by "i^{θ}" can also be indicated by "$\cos\theta + i\sin\theta$", where Cosine and Sine are working in a system that divides a circle up into 4 pieces.

- You can rephrase "i^{θ}" so that θ can be a value in degrees or radians or any other way of dividing up a circle. For example, "$i^{(\theta/90)}$" works with θ in degrees; "$i^{(2\theta/\pi)}$" works with θ in radians. It is still the case that any of these new exponentials will still be equal to "cos θ + i sin θ", where θ, Cosine and Sine are working in that particular way of dividing up a circle.

- You can rephrase these exponentials to be a base raised to an Imaginary power. For example, "$(\sqrt[180i]{-1})^{i\theta}$" works in degrees; "$(\sqrt[i\pi]{-1})^{i\theta}$" works in radians. It is still the case that these are equal to "cos θ + i sin θ", where θ, Cosine and Sine are working in that particular way of dividing up a circle.

- You can swap the bases of these exponentials for Real numbers. So, "$1.01761^{i\theta}$" works with θ in degrees; "$e^{i\theta}$" works with θ in radians. And it is still the case that these will be equal to "cos θ + i sin θ", where Cosine and Sine are working in that particular way of dividing up a circle, and θ is an angle in that system. In other words, "$1.01761^{i\theta}$ = cos θ + i sin θ", when θ is in degrees; "$e^{i\theta}$ = cos θ + i sin θ", when θ is in radians.

- Therefore, "$e^{i\theta}$" is really a way of identifying the position of a point on a unit-radius circle in terms of how much the point at "1 + 0i" would have had to have been rotated to get there. The amount of rotation is given by a multiplication by "$e^{i\theta}$".

All this means that the properties of "$e^{i\theta}$" are in no way mysterious, complicated or unique. The equivalence of "$e^{i\theta}$" to "cos θ + i sin θ" is unremarkable and to be expected.

There are ideas here that I haven't seen elsewhere – I'm fairly sure that most people don't know anything about them. Even if the ideas here aren't particularly profound, they will probably help people understand some aspects of maths better.

To get the most from the following explanation, you will need to have at least a trivial understanding of Complex numbers and exponentials, and of course you will need to have some experience of "$e^{i\theta}$". A good test for whether you will get what I am trying to say or not is if you understand how Sine and Cosine indicate the y-axis and x-axis values of points on the circumference of a unit-radius circle at particular angles from the origin. If you generally think of Sine and Cosine in this way, then this explanation should be straightforward. However, I've noticed that some people don't think of Sine and Cosine in this way at all, and see the functions as being far more complicated than they actually are. At the extreme, if you are someone who thinks it is impossible to calculate the Sine and Cosine of an angle by drawing a circle, then you will definitely struggle with this explanation.

Note:

- I am not a mathematician. Therefore, I probably explain things in a non-mathematical way, and I might use the wrong terms at times. There are probably mistakes in places.

- This entire explanation is a result of my attempt to understand waves and signal processing in a simple way. Apart from Michael Ossmann's excellent lessons on his website: https://greatscottgadgets.com/sdr, I couldn't find much on signal processing that was straightforward enough for my level of maths. Therefore, I decided to figure everything out myself instead. Ironically, if I were good at maths, I wouldn't have spent any time thinking about "$e^{i\theta}$".

- This explanation is meant to be easy to understand. Therefore, there are things that will be obvious to people who are good at maths, and also things that might be obvious to everyone. It might seem overly repetitive at times.

- The first chapter is designed to get everyone thinking in the same way.

- I sometimes include unnecessary ones and zeroes in Complex numbers in order to make the explanation clearer.

- For decimal fractions, I tend to round the value to 4 decimal places unless the number is particularly significant at that moment.

- When writing Imaginary numbers, I sometimes put the "i" before the numbers it is being multiplied by; I sometimes put it afterwards. This depends on the context and which is easiest to read at the time.

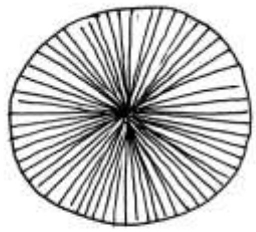
Chapter 1: Background

Ways of dividing up a circle

An angle is a value that represents a portion of a circle. How much of a circle an angle represents depends on how many pieces a circle has divided into. There are countless ways to divide up a circle, and therefore, there are countless angle systems.

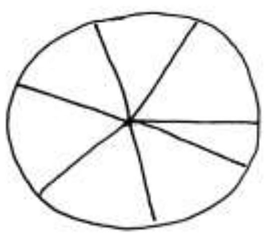
360

If we divide a circle up into 360 portions, each angle will represent one 360^{th} of a circle. Obviously, we call such angles "degrees". In this system, 90 divisions, or angles, represent a quarter of a circle; 180 divisions represent half a circle; 360 divisions represent a whole circle.



2π

If we divide a circle up into 2π portions, each angle will be one $2\pi^{th}$ of a circle. We call such angles, "radians". In this system, 0.5π divisions represent a quarter of a circle; π divisions represent half a circle; 2π divisions represent a full circle.



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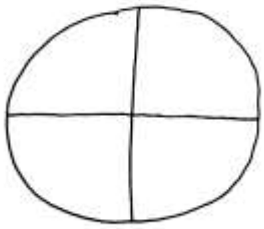
We could also divide up a circle into 4 portions. In such a system, each angle will be quarter of the circle. One division will represent a quarter of a circle; 2 divisions represent half a circle; 4 divisions represent a full circle.



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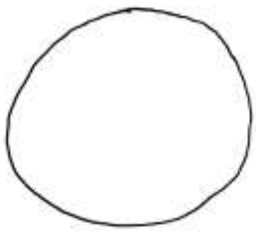
We could also divide a circle into 1 portion, not that there is really any dividing in such a case. If we do this, 1 division represents the full circle. Half a division represents half a circle; quarter of a division represents quarter of a circle.

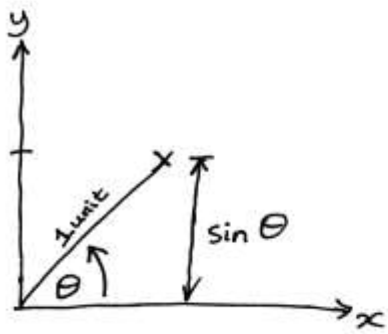


Other Systems

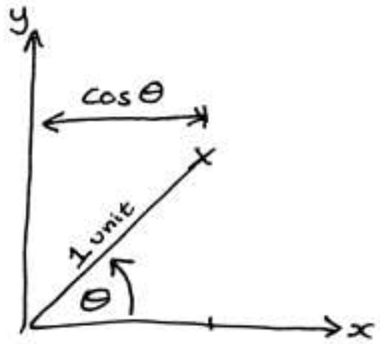
Obviously, there are countless ways to divide up a circle, but in this explanation, we will stick mainly to these four.

Sine and Cosine

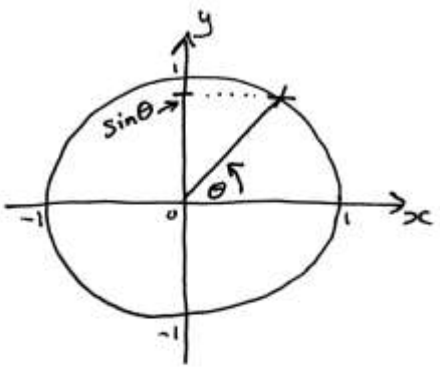
If we consider points on x and y-axes, the Sine of a value gives the height of a point that is one unit away from the origin at the angle of that value.

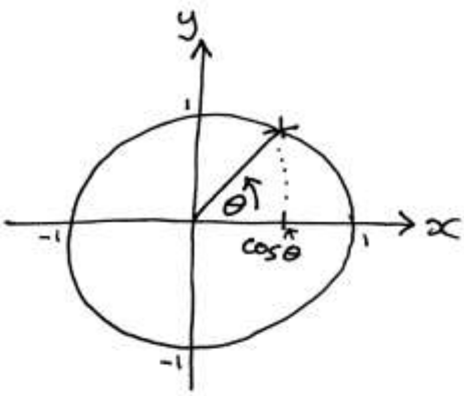


The Cosine of a value gives the horizontal distance outwards of a point one unit away from the origin at the angle of that value.



On a unit-radius circle that is centred on x and y-axes, the Sine of the angle of any point will be that point's y-axis value. The Cosine of the angle of any point will be that point's x-axis value.



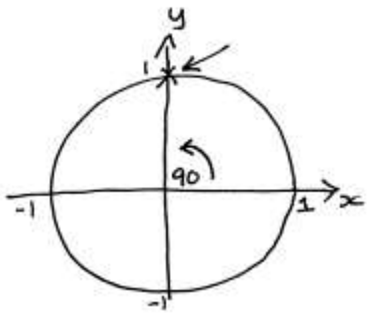


Thinking of this the other way around, *any* point on the unit-radius circle can be identified by giving its coordinates in terms of the Cosine of the angle and the Sine of the angle.

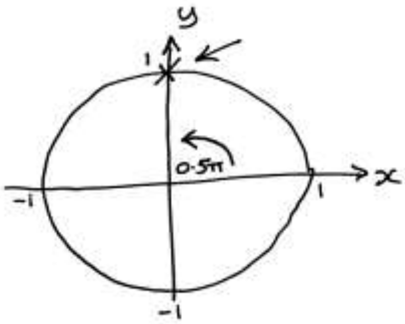
Sine and Cosine With Different Methods of Dividing up a circle

The Sine and Cosine functions give results based on the *portion* of a circle that is represented by an angle. If that portion is being measured in degrees, then the Sine and Cosine functions must be working in degrees to treat that angle as the correct portion of a circle. If that portion is being measured in radians, then the Sine and Cosine functions must be working in radians to treat that angle as the correct portion of a circle. If that portion is being measured in a system where the circle is divided into 4 parts, then the Sine and Cosine functions must be working within that system too. Given that there are countless ways to divide up a circle, there are also countless ways that the Sine and Cosine functions can operate to work with these systems. The Sine function on any particular *portion* of a circle will give the same result whether that portion is measured in degrees, radians or any other system of angles. The same is true for the Cosine function.

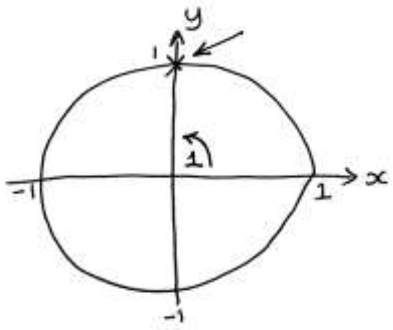
We'll now go back to the idea that any point on the unit-radius circle can be identified by giving its coordinates in terms of the Cosine of the angle and the Sine of the angle. If we are measuring angles in degrees, we can indicate a point's position by giving the coordinates in terms of the Cosine of that angle in degrees and the Sine of that angle in degrees, *when Sine and Cosine are working in degrees*. For example, the point at 90 degrees can be identified using the coordinates: (cos 90, sin 90). This reduces to the coordinates (0, 1).



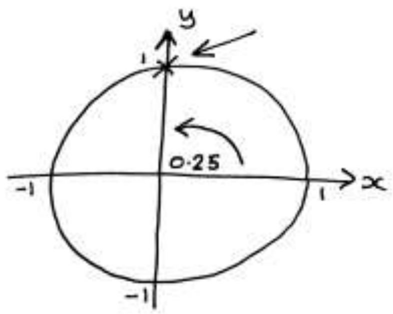
That same point, if we were using radians, would be at an angle of 0.5π radians. Its coordinates would be (cos 0.5π, sin 0.5π), *when Sine and Cosine are working in radians*, which is also (0, 1).



That same point, if we were using a system of dividing up a circle into 4 parts, would be at an angle of 1 "4-part divisions". Its coordinates would be at (cos 1, sin 1), where Cosine and Sine are working in this system of dividing a circle up into 4 parts. The coordinates would still resolve to (0, 1).



That same point, if we were using a system where the circle is in one piece, would be at an angle of 0.25 of an angle portion. Its coordinates would be at (cos 0.25, sin 0.25), where Cosine and Sine are also working in this system of the circle being in one part. The coordinates would still resolve to (0, 1).



The Complex Plane

If we draw the circle on the Complex plane, instead of giving coordinates in the form (x, y), we can use Complex numbers to describe any point on its circumference. For our point at (0, 1), we can use the Complex number: "0 + 1i".

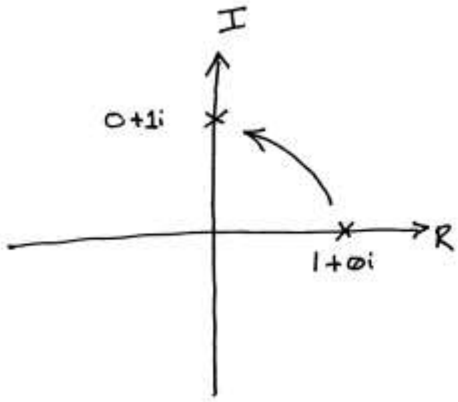
Therefore, we can use the Cosine and Sine of an angle in the form of a Complex number to identify the position of a point:

If we are using degrees, we can say our point is at "cos 90 + i sin 90".
If we are using radians, we can say that our point is at "cos 0.5π + i sin 0.5π"
If we are using the system of 4 portions in a circle, the point is at "cos 1 + i sin 1".
If we are using the system of 1 portion in a circle, the point is at "cos 0.25 + i sin 0.25".

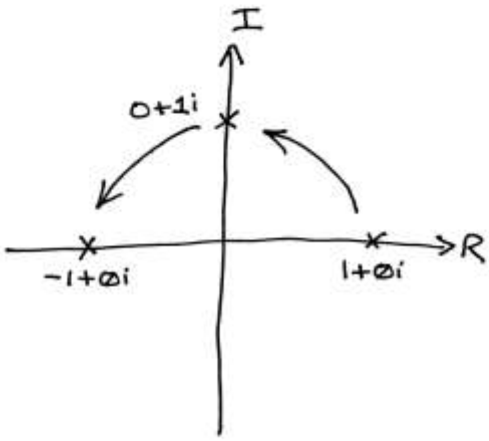
These all amount to exactly the same thing.

Multiplication on The Complex Plane

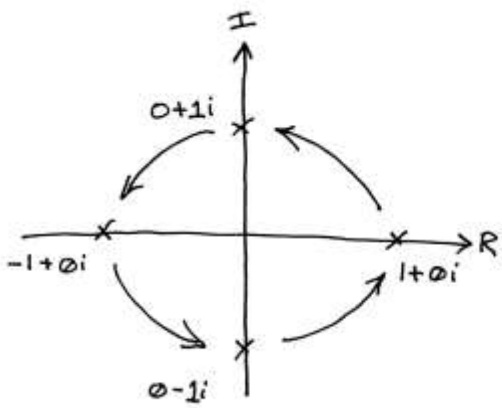
On the Complex plane, a multiplication by "i" is equivalent to a rotation of 90 degrees. Therefore, if we have a point at 1 + 0i, and multiply it by "i", we end up with a point that is the same distance from the origin, but at 0 + 1i:



If we multiply the resulting point by "i", we end up a point 90 degrees to that:



And if we multiply again by "i", we end up with a point at 270 degrees, and if we multiply again, we end up with the point at 360 degrees, which is the same as at 0 degrees.



From this we can say that:
A multiplication by “i” is equivalent to a rotation by 90 degrees.
A multiplication by “i^2” is equivalent to a rotation by 180 degrees.
A multiplication by “i^4” is equivalent to a rotation by 360 degrees.

Another way of looking at all this is that:
Multiplying by $\sqrt{-1}$ rotates by 90 degrees.
Multiplying by -1 rotates by 180 degrees.
Multiplying by +1 rotates by 360 degrees. Normally, we would say that a multiplication by 1 has no effect, but for consistency’s sake, in this case, we will say that it rotates by 360 degrees.

We can also say that a multiplication by “i^0” is equivalent to a rotation by 0 degrees. Technically, “i^0” is the same as “i^4”, but by thinking of “i^4” differently, it can help with this whole explanation.

Knowing that we can scale the exponential of “i” to achieve a particular rotation leads us to be able to perform rotations such as “$i^{0.5}$”, which will rotate a point by 45 degrees. This is the square root of “i”. The square root of *that* (i.e. “$i^{0.25}$”) will rotate a point by 22.5 degrees.

If we multiply a number by “i^3”, it would produce a rotation of 270 degrees.
If we multiplied a number by “i^8”, it would be equivalent to a rotation by 720 degrees.

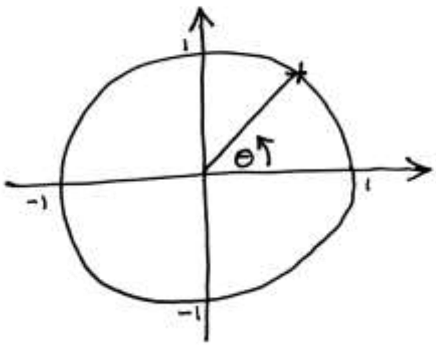
Any form of rotation, no matter how small or big can be achieved by altering the exponent of “i”.

Some Ways of Identifying a point on the unit circle

In this section, I will show some methods for identifying the position of points on a unit-radius circle's perimeter. This repeats some things I have said already.

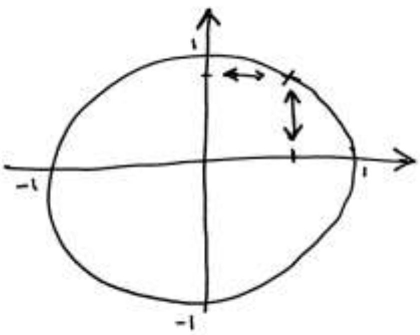
Angle

If we know that a point is on a unit-radius circle's perimeter, then we can indicate its position using just its angle from the origin:



X-axis and y-axis values

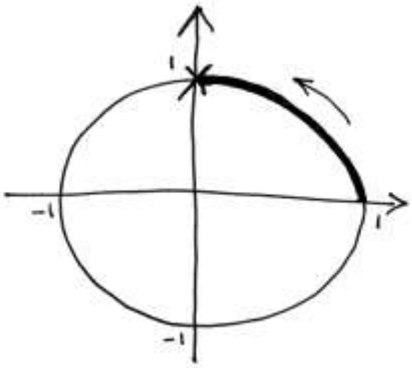
Obviously, if we have the x and y-axis values of a point's position, we know where it is, because we have its coordinates. This can also be thought of as both the Cosine of the point's angle and the Sine of that point's angle:



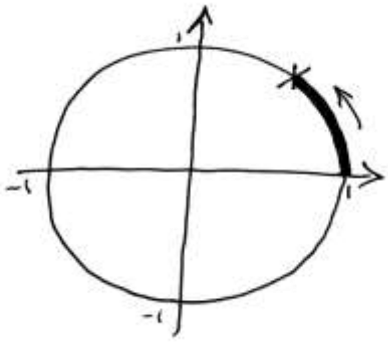
Distance Along the Circumference

We could also identify a point's position by how far along the circumference it is in relation to the "start" of a circle's edge. We will measure anticlockwise and say that the start of a circle's edge is at 0 degrees. We could call such a measure, the "Circumference Distance".

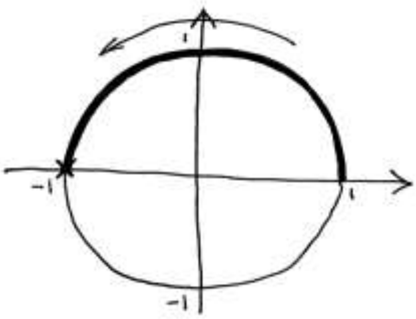
For example, if a point is a quarter of the way around the circle's edge, it will be 0.5π units along the circumference:



If a point is an eighth of the way around the circle's edge, we can give its Circumference Distance as being 0.25π units.



If a point is half way around the circle, its Circumference Distance is π units.



You might notice that the Circumference Distance is identical to the angle in radians. You could say that the angle and the Circumference Distance are really just different ways of thinking about the same thing.

Logarithmic Distance Along the Circumference

We could measure the Circumference Distance using a logarithmic scale. In this sense, the further around the circle the point was, the exponentially larger its Logarithmic Circumference Distance would be. We could also use a Logarithmic *Angle* system to match this, in the same way that radians match the non-logarithmic Circumference Distance.

Rotation Amount In Relation to the point at 0 degrees.

We could identify a point on the unit-radius circle by giving by how many angles the point at 0 degrees would have to have been rotated to get it to that position. In other words, if a point is an eighth of the way around a circle, we could say that the point at 0 degrees would have had to have been rotated 45 degrees to get it there. Although, some people would say that this is no different to giving its angle (which is true if you only consider the end product), the nuance is different. We aren’t giving the absolute angle value – instead we are specifically giving the rotational amount. They both amount to exactly the same thing, but the thinking behind them is different.

Chapter 2: i to a Power

A Point's Position in Terms of Multiplication by "i"

We can rephrase the Rotation Amount idea so it works on the Complex plane. In other words, we can identify a point's position around the unit-radius circle's perimeter by saying how much the point at "1 + 0i" would have had to have been rotated to end up at the position of our point.

Given that the value "1 + 0i" would require a different amount of rotation to get to any particular point on a unit-radius circle's edge, the position of any point can be identified solely by that particular amount of rotation.

As we saw in chapter 1, we can perform particular amounts of rotation by multiplying a number by "i" raised to an exponent. The exponent dictates how much rotation happens.

Therefore, we can *indicate* amounts of rotation using "i" raised to a power.

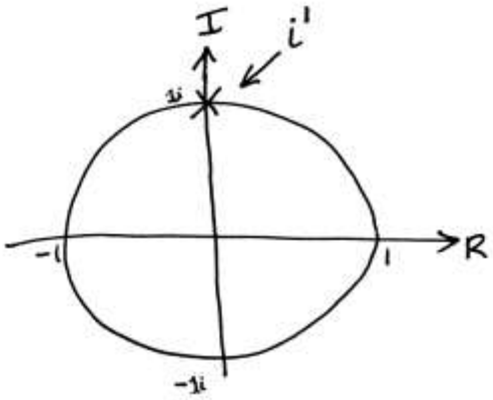
And, therefore, we can exclusively *identify* individual points around a unit-radius circle using "i" raised to particular powers.

In this sense, we are really multiplying "1 + 0i" by powers of "i", which is the same as multiplying 1 by powers of "i", which is really just the same as showing powers of "i".

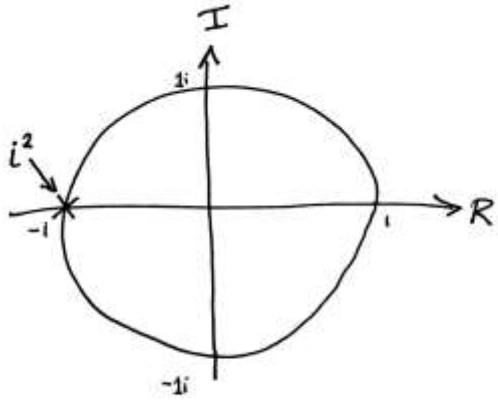
The basic rule for this idea, and the basis for this entire explanation is that:
Any point on a unit-radius circle can be identified using just "i" raised to a power.

Some examples:

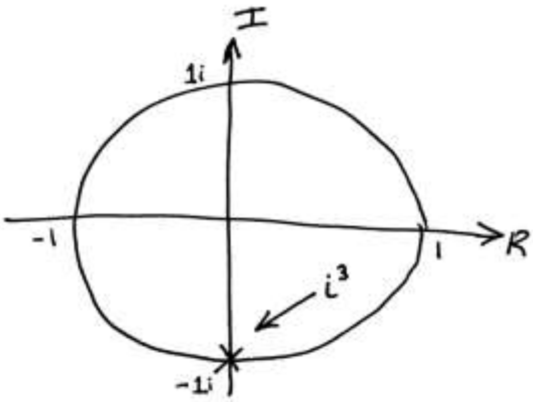
A point that is at 90 degrees on a unit circle, or in other words at "0 + 1i", can be identified by the fact that it must have taken a rotation of 90 degrees for "1 + 0i" to get there. In terms of "i", it must have been multiplied by "i^1" in order for it to be there. Therefore, just the exponential "i^1" is sufficient to identify the position of our point. No other point is represented by "i^1", so "i^1" exclusively indicates the position of our point:



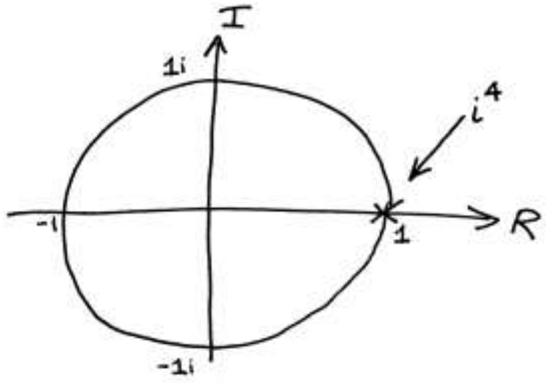
A point that is at 180 degrees can be identified by "i^2". The point "1 + 0i" would have required a rotation of 180 degrees to get there. Multiplying by "i^2" is the same as a rotation of 180 degrees.



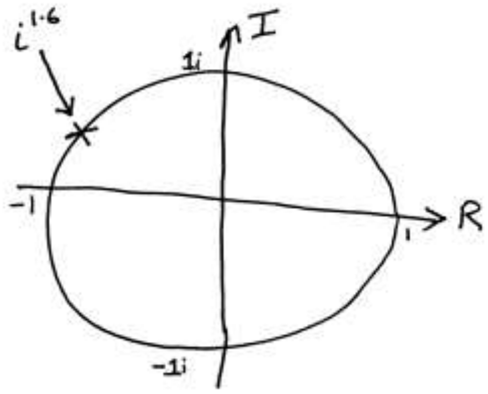
A point that is at 270 degrees can be identified by "i^3". The point at "1 + 0i" would have required a rotation of 270 degrees to get there, and this is equivalent to a multiplication by "i^3".



A point at 360 degrees can be identified by "i^0" or by "i^4". You could say that the point at "1 + 0i" would have required no rotation to end up where it started, or you could say that it would have required a rotation of 360 degrees to get there.



And, as a random example, the value "$i^{1.6}$" indicates a point that is at "-0.8090 + 0.5878i" [to 4 decimal places]. The point at "1 + 0i" would have required a rotation equivalent to a multiplication by "$i^{1.6}$" to end up at "-0.809 + 0.5878i".



Calculating "i" raised to a power.

Note that we don't need a calculator or knowledge of "$e^{i\theta}$" to calculate "i" raised to any power. We can calculate it approximately in the same way that we can calculate the Sine and Cosine of any angle – with a unit-radius circle.

Suppose we wanted to know where the point identified by "$i^{3.4}$" is (which is really the point that is reached by multiplying "1 + 0i" by "$i^{3.4}$". First, we draw a large circle on a piece of graph paper. The bigger the circle, the more accurate the final result will be. We measure the radius – this will be our "unit" around which all measurements will be based. We then measure 0.85 of these units (3.4 ÷ 4) along the circumference and mark the point there. This point marks the result of "1 + 0i" multiplied by "$i^{3.4}$". Obviously, measuring along the circumference is not easy, so it is better to use a protractor instead. In this case, we just multiply 0.85 by 360 to get 306 degrees, and then mark the point at 306 degrees.

With an 8 cm radius circle drawn on graph paper, it is easy to get the y-axis value of the point as -0.81 units and the x-axis value of the point as 0.59 units. Therefore the coordinates of this point are (0.59, -0.81). The Complex number that marks this point is therefore, "0.59 – 0.81i". We can say that "$i^{3.4}$" ≈ "0.59 – 0.81i"

We can check this with a calculator, but the calculator might have used knowledge of "$e^{i\theta}$" in its method. A calculator will give the result as "0.5878 – 0.8090i" [rounded to four decimal places].

Given that we know the point is at 306 degrees, we could have skipped drawing a circle, and instead just used Sine and Cosine on a calculator to work out the result. The Sine of 306 degrees is -0.8090; the Cosine of 306 degrees is 0.5878. Therefore, the Complex number that indicates the point is "0.5878 – 0.8090i".

Analysis of i^θ as a Form of Identification

We can see that:

"i" raised to a Real power identifies any point on a unit-radius circle's edge.

This system uses a logarithmic way of measuring around the circle. Each point further around the circle needs an exponentially higher value to mark its position. The exponential "i^4" marks the position of a point at 360 degrees. The square root of that, "i^2", marks the position of a point at 180 degrees. The square root of *that* marks the point at 90 degrees. The square root of *that* marks the point at 45 degrees and so on.

We can also see that using "i" raised to a power results in a system where the circle is divided up into 4 portions. If we say that θ is an angle in a system where a circle is divided into four portions, then i^θ gives the position of a point at that angle. Given *that,* we can say that that point's position can also be given by "cos θ + i sin θ" where the Cosine and Sine functions are also operating within a system that divides a circle up into 4 portions.

The value of θ increases in the normal way for points around the circle, but the total value of the exponential increases exponentially for points around the circle.

Other Radiuses

We can also identify points on circles that have radiuses that are of lengths *other* than 1 unit. To do this we, just scale the whole of the "i^θ" value accordingly.

If we want to indicate the position of a point on a circle with a radius of 0.5 units, we can just give the position as: 0.5 * (i^θ), or $0.5i^\theta$.

If we want to indicate a point on a circle with a radius of 111 units, we can just give its position as $111i^\theta$.

Note that, strictly speaking, we are not really *scaling* the result of "i^θ" by a chosen value, but just *rotating* that chosen value (treated as a Complex number at 0 degrees) by the amount dictated by a multiplication by "i^θ". Therefore, the 2 in "$2i^\theta$" is really the Complex number "2 + 0i", or in other words, the point on a 2 unit-radius circle at 0 degrees. In the formula "$2i^\theta$", we are rotating "2 + 0i" by the amount dictated by a multiplication by "i^θ". This obviously amounts to the same thing, but it's easier to visualise this explanation if you think of it this way.

Anyway, what this means is that we can actually identify any point in the Complex plane by using Real numbers multiplied by exponentials with "i" as the base.

Therefore, we have come up with a way of identifying any point on the Complex plane using an exponential, but in a way that is straightforward and intuitive. Hopefully anyone with a slightly technical mind can understand this. This method doesn't require any huge leaps of faith.

Other Types of Angles

So far, the θ in "i^θ" is an angle in a system where the circle is divided into 4 parts. We can, of course, adjust the formula so we can have θ as an angle in other systems.

1 Piece Circles

If we want to use the "i^θ" formula in circles that are in 1 piece (in other words circles that have not been divided up), we will need to multiply the θ by 4 before we subject it to being an exponent of "i". Therefore, the position of any point on the unit-radius circle can be given as: "$i^{4\theta}$", where θ is a fraction of a circle.

Examples of this in use are:
If the point is 0.25 of the way around the circle, its position is "$i^{4*0.25}$" = "i^1" = 0 + 1i.
If the point is 0.5 of the way around the circle, its position is "$i^{4*0.5}$" = "i^2" = -1 + 0i.

In this system, "$i^{4\theta} = \cos\theta + i \sin\theta$", when θ is a fraction of a circle, and Cosine and Sine are operating in a system where a circle is in one piece.

Circles Divided into 360 Pieces

If we want to use the "i^θ" formula in circles that have been divided into 360 pieces (in other words with degrees), we need to multiply the θ by 4, and then divide that by 360 before we subject it all to being an exponent of "i". Therefore, the position of any point on the unit-radius circle can be given as: "$i^{(4\theta/360)}$" = "$i^{(\theta/90)}$", where θ is an angle in degrees.

Examples of this in use are:
If the point is at 45 degrees, its position is "$i^{45/90}$" = "$i^{0.5}$" = 0.7071 + 0.7071i.
If the point is at 252 degrees, its position is "$i^{252/90}$" = "$i^{2.8}$" = -0.3090 – 0.9511i.

In this system, "$i^{(\theta/90)} = \cos\theta + i \sin\theta$", when θ is an angle in degrees, and Cosine and Sine are operating in degrees.

Circles Divided into 2π Pieces

If we want to use the "i^θ" formula in circles that have been divided into 2π pieces (in other words with radians), we need to multiply the θ by 4, and then divide that by 2π before we subject it all to being an exponent of "i". Therefore, the position of any point on the unit-radius circle can be given as: "$i^{(4\theta/(2\pi))}$" = "$i^{(2\theta/\pi)}$", where θ is an angle in radians.

Examples of this in use are:
If the point is at 0.25π radians, its position is "$i^{(2 * 0.25\pi)/\pi}$" = "$i^{0.5}$" = 0.7071 + 0.7071i.
If the point is at 1.23π radians, its position is "$i^{(2 * 1.23\pi)/\pi}$" = "$i^{2.46}$" = -0.7501 – 0.6613i.

In this system, "$i^{(2\theta/\pi)} = \cos\theta + i \sin\theta$", when θ is an angle in radians, and Cosine and Sine are operating in radians.

Chapter 3: Changing the Bases

At the moment, our formulas for describing the position of a point around a unit-radius circle are as follows:
For a circle divided into 4 parts: "i^{θ}".
For a circle as one part: "$i^{4\theta}$".
For a circle in degrees: "$i^{\theta/90}$".
For a circle in radians: "$i^{2\theta/\pi}$".
... where the θ in each formula refers to an angle only for that specific way of dividing up a circle. In other words, each θ in this list is representing a different type of angle.

We could of course have formulas for countless other ways of dividing up a circle.

We can change each of these formulas so that instead of them being "i" raised to a power, they become a value raised to a multiple of "i". This is actually the same as that value raised to the power of "i", and then that exponential being raised to the power of a Real number. The actual meaning of the exponentials and what they do will be identical. All we are doing is changing how the calculations are formed.

The basic idea here is that we convert:
"$i^{\theta?}$" to "$b^{i\theta}$"
...where θ represents an angle in a particular system of dividing up a circle, "?" represents some value multiplying or dividing θ, and "b" is the new base for the exponential.

Doing this will reduce the complexity of the exponent (not that it was particularly complicated anyway), but result in a more complicated exponential. The significant problem in doing this is that we lose the straightforwardness of what the exponential means. We go from something we can easily visualise, to something that, as far as I can tell, few people can visualise without knowing about this step.

Circle Divided Into 4 Parts

We'll start with a circle divided into 4 parts and the simple formula: "i^{θ}".

We want to find the value "b" in the following:
$b^{i\theta} = i^{\theta}$

To do this, we'll make both halves refer to an actual point around the unit circle. We'll use the point at i^1, which is a quarter of the way around the circle. There is no particular reason for me picking this point over any other. Therefore, we have:
$b^i = i$

This can be written as:
$b = \sqrt[i]{i}$
... or in words, "b" is equal to the i^{th} root of i.

We can say that for a circle that is divided into 4 portions, and where θ is an angle in that system, $\sqrt[i]{i}$ raised to the power of iθ will indicate the position of a point on a unit-radius circle at an angle of θ. Furthermore, that point's position can also be indicated by "cos θ + i sin θ", where Cosine and Sine are working within a system that divides a circle into 4 parts. Or, to put this another way:

$(\sqrt[i]{i})^{i\theta} = \cos\theta + i\sin\theta$, where θ is an angle based on a system that divides a circle into 4.

Note that the above equivalence is nothing new. We know that "$i^\theta = \cos\theta + i\sin\theta$" when θ is an angle based on a system that divides a circle up into 4 parts, and when Sine and Cosine are working in that system. Therefore, it should be obvious that *any* formula that means the same thing as "i^θ", no matter how it is phrased, will *always* be equivalent to "cos θ + i sin θ" in a 4 division angle system. Similarly, if we alter the "i^θ" formula so that θ can be used as representing angles in other angle systems, then that adjusted formula will *always* be equivalent to "cos θ + i sin θ" in that other angle system.

To illustrate the base being used to represent points on a unit-radius circle in practice, here are some examples. These are not the most thoroughly representative examples, but they illustrate the concept simply.

Example 1:

The point at 2 angle units on a unit-radius circle divided into 4 parts can be represented with: $(\sqrt[i]{i})^{2i}$.

This resolves to:
$(i)^2 = -1$.
The point's position is at -1 + 0i.

Example 2:

The point at 4 angle units can be represented with: ($\sqrt[i]{i}$)^{4i}.

This resolves to:
(i)^4 = (-1)^2 = 1.
The point's position is at 1 + 0i.

Example 3:

The point at 0.5 angle units can be represented with: ($\sqrt[i]{i}$)$^{0.5i}$.

This resolves to:
(i)$^{0.5}$
The point's position is at 0.7071 + 0.7071i.

Circle With 1 Part

To rebase the formula "$i^{4\theta}$", we need to solve this:
$b^{i\theta} = i^{4\theta}$

To solve this, we will use the value of θ that represents the equivalent of 180 degrees – in other words, 0.5. Again, the choice of this number is completely arbitrary. We end up with this:
$b^{0.5i} = i^{4*0.5}$
$b^{0.5i} = i^2$
$b^{0.5i} = -1$
b = $\sqrt[0.5i]{-1}$

If we had used, say, 0.25, we would have ended up with this:
$b^{0.25i} = i^{4*0.25}$
$b^{0.25i} = i^1$
$b^{0.25i} = i$
b = $\sqrt[0.25i]{i}$
... which amounts to exactly the same thing.

If we had used 1 to find the base, we would have ended up with $\sqrt[i]{1}$, which isn't as helpful as a value.

We'll say that the base is $\sqrt[0.5i]{-1}$.

We can identify the position of any point around the unit circle by giving $\sqrt[0.5i]{-1}$ raised to the power of iθ, where θ is an angle in a system of angles where the circle is "divided" into 1 portion. In other words, if θ is a fraction of a circle, then $(\sqrt[0.5i]{-1})^{i\theta}$ identifies the position of a point on the unit circle's edge at that fraction of the way around the circle. As always, it is still the case that that point's position can also be indicated by "cos θ + i sin θ", where Cosine and Sine are working within a system where θ is a fraction of the way around the circle. As I said before, this is expected. Anyway, we can phrase this equivalence like so:
$(\sqrt[0.5i]{-1})^{i\theta}$ = cos θ + i sin θ, where θ is a fraction of the way around a circle, and Cosine and Sine are operating in a system where the circle is in one piece.

Example 1:

The point at 0.5 angle units on a unit-radius circle "divided" into 1 part can be represented with:
$(\sqrt[0.5i]{-1})^{i0.5}$.
This resolves nicely to -1, which means the point is at -1 + 0i.

Example 2:

The point at 0.25 angle units can be represented with:
$(\sqrt[0.5i]{-1})^{i0.25}$.
This resolves to $\sqrt{-1}$, or "i", so the point is at 0 + 1i.

Example 3:

The point at 0.75 angle units can be represented with:
$(\sqrt[0.5i]{-1})^{i0.75}$.

This resolves as:
$(\sqrt[0.5]{-1})^{0.75}$.
$= (\sqrt[2]{-1})^{3}$.
$= (i)^{3}$
= i * i * i
= -1 * i
= -i.

Therefore, the point is at 0 – 1i.

Circle Divided Into 360 Parts

To rebase the formula "$i^{\theta/90}$", we need to solve this:
$b^{i\theta} = i^{\theta/90}$

To solve this, we will use the value of θ that represents the equivalent of 180 degrees, which obviously, is 180 degrees. Therefore, we end up with this:
$b^{180i} = i^{180/90}$

This resolves to:
$b^{180i} = i^2$
$b^{180i} = -1$
$b = \sqrt[180i]{-1}$

We can use this as a base to the power of iθ where θ is an angle based on a system that divides a circle into 360 portions. To put this in more concise words, $(\sqrt[180i]{-1})^{i\theta}$ gives the position of a point on a unit-radius circle at an angle of θ, where θ is an angle in degrees. As always, it is still the case that the point's position can be given by "cos θ + i sin θ" when θ is in degrees. Or, to put this another way:
$(\sqrt[180i]{-1})^{i\theta} = \cos\theta + i \sin\theta$, when θ is in degrees, and Cosine and Sine are operating in degrees.

Example 1:

The point at 180 degrees can be represented with:
$(\sqrt[180i]{-1})^{i180}$

This resolves to -1. Therefore, the point is at -1 + 0i.

Example 2:

The point at 270 degrees can be represented with:
$(\sqrt[180i]{-1})^{i270}$

This resolves to:
$(\sqrt[2]{-1})^3$
$= (i)^3$
$= -i$.

Therefore, the point is at 0 - 1i.

Example 3:

The point at 22.5 degrees can be represented with:
$(\sqrt[180i]{-1})^{i22.5}$

This resolves to:
$(\sqrt[180 \div 22.5]{-1})$
$= \sqrt[8]{-1}$
$= \sqrt[4]{i} = i^{0.25}$

Using our old system of identifying points around a circle using "i" raised to powers, we know that $i^{0.25}$ is at the equivalent of 22.5 degrees [which we also knew because that's the angle we started with in this example], so the point is at "cos 22.5 + i sin 22.5", where 22.5 is a value in degrees. This works out as "0.9239 + 0.3827i".

Circle Divided Into 2π Parts

To rebase the formula "$i^{2\theta/\pi}$", we need to solve this:
$b^{i\theta} = i^{2\theta/\pi}$

To solve this, we will use the value of θ that represents the equivalent of 180 degrees, which is the angle of π. We end up with this:
$b^{i\pi} = i^{(2*\pi)/\pi}$

This resolves as:
$b^{i\pi} = i^{(2\pi)/\pi}$
$b^{i\pi} = i^{2}$
$b^{i\pi} = -1$
$b = \sqrt[i\pi]{-1}$

We can use this as a base to the power of iθ where θ is an angle based on a system that divides a circle into 2π portions, or in other words where θ is in radians. The formula: $(\sqrt[i\pi]{-1})^{i\theta}$ gives the position of a point on a unit-radius circle at an angle of θ, where θ is an angle in radians. As always, that point can also be identified using "cos θ + i sin θ" when θ is in radians, and as always, this is to be expected.
$(\sqrt[i\pi]{-1})^{i\theta} = \cos\theta + i \sin\theta.$

It is probably worth clarifying here that all these equivalences are *only* true for the particular angle system for which the base is chosen. Therefore, it is not the case that the above equation is true for any angle system apart from radians. If the circle is divided up into, say, 360 portions, then the above equation will not work.

Example 1:

The point on a unit-radius circle at an angle of π radians is situated at: $(\sqrt[i\pi]{-1})^{i\pi}$.

This resolves nicely to -1, which is the point at -1 + 0i.

Example 2:

The point at an angle of 0.5π radians is situated at: $(\sqrt[i\pi]{-1})^{0.5\pi i}$.

This resolves to:
$(-1)^{0.5}$
$= \sqrt{-1}$
= i.

... which is the point at 0 + 1i.

Example 3:

The point at an angle of 1.75π radians is situated at: $(\sqrt[i\pi]{-1})^{1.75\pi i}$.

This resolves to:
$(-1)^{1.75}$
$= (\sqrt{-1})^{3.5}$
$= i^{3.5}$

From thinking of our original circle system with “i” raised to a power, we can see that this will be: 0.7071 - 0.7071i.

Chapter 4: Changing the Bases to Real Numbers

Our original exponentials with "i" as a *base* were:
For a circle divided into 4 parts: "i^{θ}".
For a circle as one part: "$i^{4\theta}$".
For a circle in degrees: "$i^{\theta/90}$".
For a circle in radians: "$i^{2\theta/\pi}$".

Our new exponentials with "i" as an *exponent* are:
For a circle divided into 4 parts: "$(\sqrt[i]{i})^{i\theta}$".
For a circle as one part: "$(\sqrt[0.5i]{-1})^{i\theta}$".
For a circle in degrees: "$(\sqrt[180i]{-1})^{i\theta}$".
For a circle in radians: "$(\sqrt[i\pi]{-1})^{i\theta}$".

With no actual evidence, we'll presume that these new bases can all be converted into Real numbers, which is something that would make them a lot easier to use. We just have to find a way to do this.

One fact about all of these exponentials is that they can all be used to refer to the same points around the circle. Therefore, they can all be equivalent if the appropriate values are fed into them that refer to the same point.

For example, if we put the equivalent of an angle of 180 degrees into each exponential formula, they will all result in -1 + 0i, which is -1:
$(\sqrt[i]{i})^{i2} = -1$
$(\sqrt[0.5i]{-1})^{0.5i} = -1$
$(\sqrt[180i]{-1})^{i180} = -1$
$(\sqrt[i\pi]{-1})^{i\pi} = -1$
... which means that all of *these* formulas are the same as each other. However, (as far as I can tell) this doesn't help solve anything because all the formulas were derived from the same original formula. Using these will at best mean we end up with a conclusion such as "1 = 1".

If we put the equivalent of an angle of 360 degrees into each formula, all the formulas will result in 1 + 0i, which is 1:
$(\sqrt[i]{i})^{i4} = 1$
$(\sqrt[0.5i]{-1})^{i} = 1$
$(\sqrt[180i]{-1})^{i360} = 1$
$(\sqrt[i\pi]{-1})^{i2\pi} = 1$
... which means that all of these are the same as each other too.

The significant idea about the last set is that in each case we have an exponential that is equal to the full circle. For this to be so, it means that the smaller the base, the larger the exponent must be, and the larger the base, the smaller the exponent must be. There must be some kind of inverse relationship between the bases and the exponents for them all to equal the same amount.

Given *that*, and given the fact that we are presuming the bases can all be expressed with Real numbers: if we remove the “i” from the exponents, the formulas would all be equal to the same Real number. We will call this number “S” for “The Special Number”. In other words:
$(\sqrt[i]{i})^4 = S$
$(\sqrt[0.5i]{-1})^1 = S$
$(\sqrt[180i]{-1})^{360} = S$
$(\sqrt[i\pi]{-1})^{2\pi} = S$
... where S is the same Real number in each case.

This means that the number S in the Real world is an equivalent to the full circle in the Complex plane. It is the value that links an exponential with a Real base and Real exponent to a circle represented by a Real base and an Imaginary exponent.

Given that S represents a full circle, and that S is calculated with exponential numbers, it will be the case that $\sqrt{S}$ will be the equivalent of half a circle (i.e. 180 degrees), $\sqrt[4]{S}$ will be quarter of a circle (i.e 90 degrees), $\sqrt[8]{S}$ will be an eighth of a circle (i.e. 45 degrees), and so on.

This means that any exponential with a Real base and a Real exponent, that is equal to, say, $\sqrt{S}$, will, if the exponent is multiplied by “i”, be equal to a point on the Complex plane that marks the position of a point on a unit-radius circle at 180 degrees.

In other words:
If $b^c = S$, then b^{ic} = the position of a point on a unit-radius circle at an angle of 360 degrees.
If $b^c = S$, then $b^{0.5ic}$ (in other words, $\sqrt{b^{ic}}$) = the position a point on a unit-radius circle at an angle of 180 degrees.
If $b^c = S$, then $b^{0.75ic}$ = the position of a point on a unit-radius circle at an angle of 270 degrees.

The Use Of S

If we can find S, we will be able to calculate the bases for all our new exponentials, and for any other exponentials we care to have, and with a minimum of effort.

We know that:
$b^c = S$
...where b is a base, and c is the number of divisions in a circle. Therefore, if we knew what S is, we would be able to find the base for any system of dividing up a circle using this equation:
$b = \sqrt[c]{S}$
... where "c" represents the desired divisions in a circle, and "b" is the base that will produce an exponential, with an Imaginary exponent, that will operate according to that angle system.

The Difficulty In Finding S

We know that:
$(\sqrt[i]{i})^4 = S$
$(\sqrt[0.5i]{-1})^1 = S$
$(\sqrt[180i]{-1})^{360} = S$
$(\sqrt[i\pi]{-1})^{2\pi} = S$

Therefore, we know that S is exactly $\sqrt[0.5i]{-1}$. However, this is no use to us as we can't find the i^{th} root of -1 using the information that we have.

We also know that S is $(\sqrt[i\pi]{-1})^{2\pi}$. This comes from the formula for a circle divided up into 2π portions: $(\sqrt[i\pi]{-1})^{2\pi i}$.

It is pretty obvious that the base to the power of 2π will be the number "e", but unfortunately, I don't know a way of working this out without actually knowing it beforehand. If you guess in advance that "e" is the base, it is possible to test that it is correct. However, if you don't know that "e" is the base, there doesn't seem to be any way (that I can think of) of knowing what the base is. Anyway, if we use "e" as the base, everything else will fall into place.

The reason I said earlier that we can't solve a calculation such as $\sqrt[0.5i]{-1}$ is because all the methods that I have seen for solving such a thing involve prior knowledge of $e^{i\theta}$. Therefore, it would be cheating at this stage to use those methods when we haven't shown that "e" is the base yet.

Chapter 5: Proof that e is the base for radians

In this chapter, I will show that "e" is the base for the exponential, "$b^{i\theta}$", where "θ" is an angle in radians. I only know how to do this by guessing in advance that "e" is correct, and showing how "$e^{i\theta}$" fits the properties we need. The first step in doing this is calculating "e^i".

Compound Bank Interest

The simplest way of calculating "e^i" is by using the concept of compound bank interest. Compound interest is when you have the interest for your bank balance paid into the account, and then the next calculation of interest is based on the new total.

If your annual interest rate is 1%, and it is all paid at the end of the year, then you will end up getting paid 0.01 times what you started with. The new balance will be 1.01 times what you started with. If you persuade your bank to pay you a twelfth of 0.01 every month, but still with compound interest, you will get paid 0.01÷12 times your balance after the *first* month, leaving you a new balance of 1 + (0.01÷12). In the second month, the same fraction of interest, 1 + (0.01 ÷ 12), will be calculated, but on your *new* balance. Therefore, you will receive slightly more than if the interest were paid in one go at the end of the year. Supposing you had £100 in your account at the start of the year, after one year with just one annual interest payment, you would have a balance of £101. After one year, with monthly compound interest, you would have:
$100 * ((1 + (0.01÷12))^{12}$ = £101.004596

If you persuaded your bank to pay you interest every day, you would get slightly more interest over the year. If you persuaded your bank to pay you every second, you would only get the tiniest bit more. Having more intervals produces diminishing returns. There is a limit. You can test this using the following formula:
Total = $a * (1 + (f ÷ n))^n$
... where:
a is the original amount in the bank account.
f is the fraction of interest for one overall period e.g. 0.01 for 1%.
n is the number of payments made in one overall period.

We can find the nature of the limit if we just focus on this part of the formula:
$(1 + (f ÷ n))^n$
If we pick a *very* large value of "n", then the formula for any one fraction of interest will result in a fairly accurate value of the limit for that fraction. The larger "n" is, the

more accurate the result. If we set "f" to 1, we end up with an approximation of the number 2.718281828459 – in other words "e". If "f" is any other value, then the result is an approximation of 2.718281828459 raised to the power of "f".

For example:
$(1 + (1 \div 1{,}000{,}000))^{1{,}000{,}000}$ = 2.7183 [to 4 decimal places]. This is approximately e^1.
$(1 + (2 \div 1{,}000{,}000))^{1{,}000{,}000}$ = 7.3890 [to 4 decimal places]. This is approximately e^2.
$(1 + (3 \div 1{,}000{,}000))^{1{,}000{,}000}$ = 20.0854 [to 4 decimal places]. This is approximately e^3.

Given all this, we can work out "e^i" by sticking "i" in the formula. To make this easier to understand, we will put in "1i".
$(1 + (1i \div 1{,}000{,}000))^{1{,}000{,}000}$
Which resolves to:
$(1 + 0.000001i)^{1{,}000{,}000}$

This just means that we multiply (1 + 0.000001i) by itself 1,000,000 times. This doesn't require any ingenious mathematical skills to do, but it is obviously quicker to use a calculator.
The result is roughly: 0.5404 + 0.8415i

Therefore, we can say that "e^{1i}" = 0.5404 + 0.8415i.

When using Cosine and Sine in the system of dividing a circle up into 2π divisions, the inverse Cosine of 0.5404 is approximately 1 radian; the inverse Sine of 0.8415 is approximately 1 radian. Therefore, we can see that "e^{1i}" gives the position of a point at an angle of 1 radian on a unit-radius circle's perimeter.

Therefore, we can say that "$e^{1i} = \cos 1 + i \sin 1$", when the value 1 is in radians, and Cosine and Sine are working in a system that divides a circle up into 2π divisions.

We can test countless multiples of "i" as exponents of "e", and they will all work in the same way. From this we can say that "$e^{i\theta} = \cos\theta + i \sin\theta$" in radians, and therefore "e" is the base that we are looking for.

Note that I could have explained all of this using the various Taylor series for "e", Sine, and Cosine, and we would have come to the same conclusion. However, I have noticed that some people treat the way that the Taylor series for "e" reveals the connection between "$e^{i\theta}$" and "$\cos\theta + i \sin\theta$" as if it were something to do with the Taylor series itself, rather than a fact of reality. The Taylor series is just *revealing* the connection – the connection isn't a result of the Taylor series.

Chapter 6: Calculating and Using "S".

We have seen that $e^{i\theta}$ can identify points around a unit-radius circle that are at an angle "θ" that is in a system of angles where the circle is divided up into 2π portions.

Therefore, the missing base for circles that divide a circle up into 2π portions is "e". We can now say that $\sqrt[i\pi]{-1}$ is equal to "e".

We can also now calculate S. The number S is the result of $e^{2\pi}$, which is 535.491655524765 [to 12 decimal places].

Note that there is nothing necessarily special about "$e^{2\pi}$" in this situation. The breakthrough in calculating S was through finding another way to calculate a base. In this case, we knew a way of calculating "e^{i}", and so that let us calculate S. If there were some well known formula for calculating the base for circles divided up into other numbers of pieces (e.g. 360 pieces, 4 pieces, 112 pieces etc), then that would have worked just as well. The number 535.4917 could be said to be equal to countless other exponentials, and it is a matter of choice that we say it is "$e^{2\pi}$" as opposed to anything else. [Although, on the other hand, "e" and "π" are probably the only number pair for which we already have names and symbols, and they have an advantage in that their values are usually programmed into calculators. Similarly, the letter "e" appears on most computer keyboards (even those in other alphabets), and the letter "π" is easily found in word processing software].

From knowing S, we can calculate the bases for our other circle formulas using the equation:
b = $\sqrt[c]{S}$.

4 Divisions

A circle that has been divided into 4 portions will have the base:
$\sqrt[4]{535.4917}$ = 4.810477380965 [to 12 decimal places].

Therefore, $4.8105^{i\theta}$ gives the position of a point on the unit-radius circle at an angle of θ, where θ is an angle in a system that divides a circle up into 4 portions.

We can also say that $4.8105^{i\theta}$ = cos θ + i sin θ, when Cosine and Sine are operating in a system that divides a circle up into 4 portions.

We can also say that:
$\sqrt[i]{i}$ = 4.8105 = $\sqrt[4]{535.4917}$ = $\sqrt[4]{e^{2\pi}}$

Looking back at our first system of indicating points around a unit-radius circle using "i^θ", we can say that "i^θ" = $4.8105^{i\theta}$

1 Division

A circle that has been "divided" into 1 portion will have 535.4917 as its base.

Therefore, $535.4917^{i\theta}$ gives the position of a point on the unit-radius circle at an angle of θ, where θ is a fraction of the way around the circle.

We can also say that $535.4917^{i\theta}$ = cos θ + i sin θ, when Cosine and Sine are operating in a system where the circle is in one piece.

We can also say that:
$\sqrt[0.5i]{-1}$ = 535.4917 = $e^{2\pi}$ = 4.8105^4

360 Divisions

A circle that has been divided into 360 portions will have the base:
$\sqrt[360]{535.4917}$ = 1.017606491206 [to 12 decimal places].

Therefore, $1.0176^{i\theta}$ gives the position of a point on the unit-radius circle at an angle of θ, where θ is an angle in degrees.

We can also say that $1.0176^{i\theta}$ = cos θ + i sin θ, when Cosine and Sine are operating in degrees.

We can also say that:
$\sqrt[180i]{-1}$ = 1.0176

and:
1.0176^{360} = 535.4917^1 = $e^{2\pi}$ = 4.8105^4

2π Divisions

As we already know, a circle that has been divided into 2π portions will have as its base: $\sqrt[2\pi]{535.4917}$ = 2.718281828459 or "e".

Therefore, $e^{i\theta}$ gives the position of a point on the unit-radius circle at an angle of θ, where θ is an angle in radians.

We can also say that $e^{i\theta} = \cos\theta + i\sin\theta$, when Cosine and Sine are operating in radians.

Other Divisions

There are countless ways to divide up a circle, and therefore there are countless exponentials that can indicate the position of points around a unit-radius circle. These can all be calculated using the formula: $b = \sqrt[c]{S}$, where "b" is the base we want to calculate, "c" is the number of portions that the circle has been divided into, and S is 535.4917. We can also calculate the number of circle divisions that a particular base will result in, by using: $c = \log_b S$.

On the next page is a table of a few examples. A value taken from first column, when raised to the power of the corresponding value from the second column, will result in 535.4917. The same exponential with the exponent multiplied by "i" will indicate a point on a unit-radius circle at an angle that is equivalent to 360 degrees.

"b":Base for a particular circle division	**"c": Number of divisions in the circle**	**Comments**
535.491655524765	1	
23.140692632779	2	This base is "Gelfond's constant".
10.089090581019	2.718... (e)	
10	2.728752707684	
8.120527396670	3	
7.389056098931	3.14... (π)	This base is e^2. This base is obvious if you remember that $(e^2)^\pi = e^{2\pi}$.
4.810477380965	4	
4.304530324517	4.304530324517	The number of divisions in the circle is the same as the base. $4.304530324517^{4.304530324517i}$ indicates a point on a unit-radius circle's edge at the equivalent of 360 degrees, in a system that divides a circle up into 4.304530324517 pieces.
3.513585624286	5	
3.14... (π)	5.488792932594	
2.718281828459 (e)	2π (6.28...)	This is $e^{2\pi}$.
2	9.064720283654	
1.874456087585	10	
1.520256423770	15	
1.369107777062	20	
1.064847773329	100	
1.017606491206	360	This is the base for working in degrees.
1.006302965923	1000	
1	[This would be infinite]	

There might be well known number pairs within the possible bases and circle divisions. If so, this would mean that they also have some connection to "e" and π.

4.304530324517

One number that stands out is 4.304530324517:
$4.304530324517^{4.304530324517} = S$
$4.304530324517^{4.304530324517i}$ indicates the position of a point at the equivalent of 360 degrees on a unit-radius circle.

I couldn't find any reference to 4.304530324517 anywhere else, but that might be because it's not significant anywhere except in this sort of explanation.

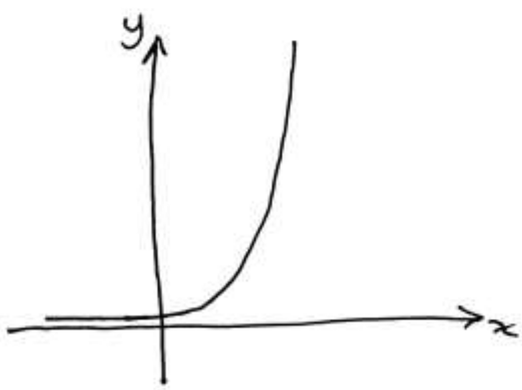
Negative Divisions

The formula "$b = \sqrt[c]{S}$" still works for negative values of "c". Therefore, it is possible to divide a circle up into a negative number of divisions, and still have a valid exponential. For example, if we want the circle divided up into -2 divisions, then the base we need is: 0.043213918264 [to 12 decimal places]. From this we can say that $0.043213918264^{-2} = S$. Furthermore, 0.043213918264^{-2i} represents a point on a unit-radius-circle that is at an angle equivalent to 360 degrees.

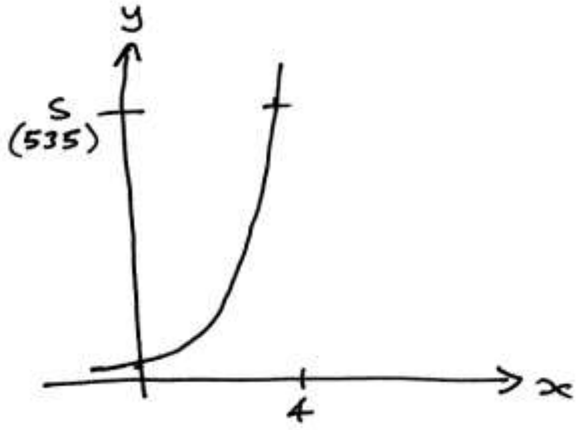
The base for a negative number of circle divisions is the reciprocal of the base for that same number of circle divisions if it were positive.

"S" On a Graph

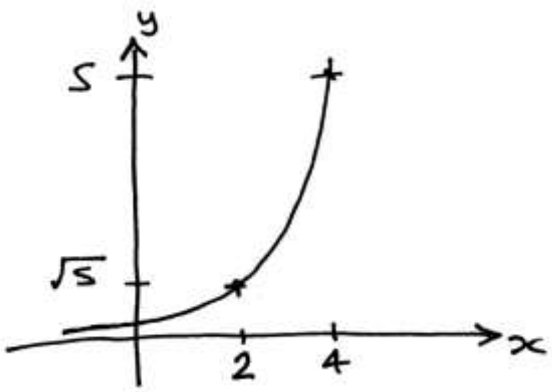
The connection between S and the circle on the Complex plane can be made more intuitive by drawing graphs. We'll take the base for a circle divided up into 4 parts: 4.810477380965. This number to the power of just x, looks like this on a graph:



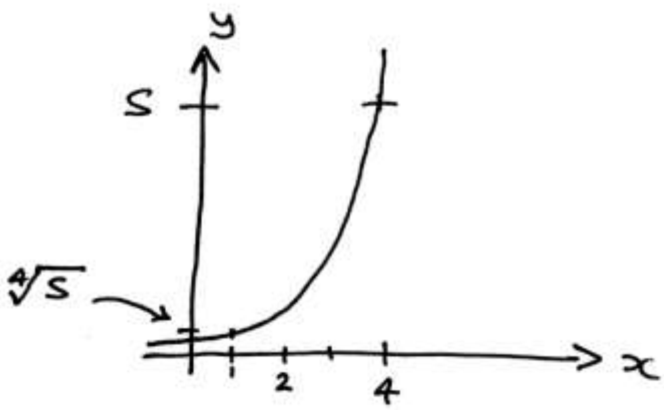
When the y-axis value is S, the x-axis value is 4, which is the number of parts that the circle has been divided into.



When the x-axis value is 2, the y-axis value is $\sqrt{S}$, which is 23.140692632779 [to 12 decimal places].

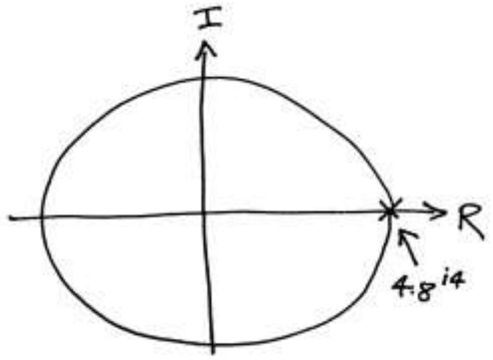


When the x-axis value is 1, the y-axis value is $\sqrt[4]{S}$, which is 4.810477380965 [to 12 decimal places].

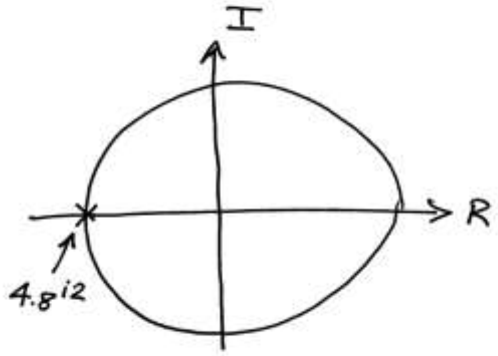


We will now look at points on the circumference of a unit-radius circle on the Complex plane, the position of which are indicated using $4.810477380965^{i\theta}$:

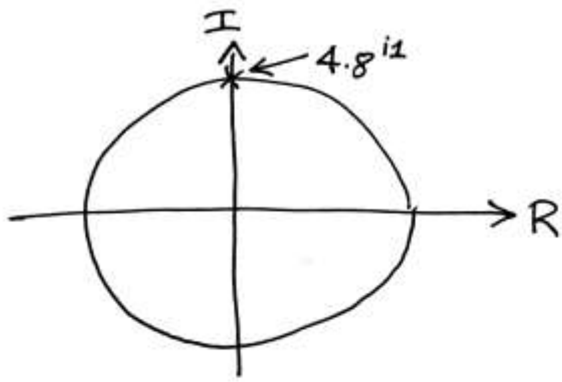
When x is 4 (i.e. when we have 4.810477380965^{i4}) , the point is all of the way around the circle:



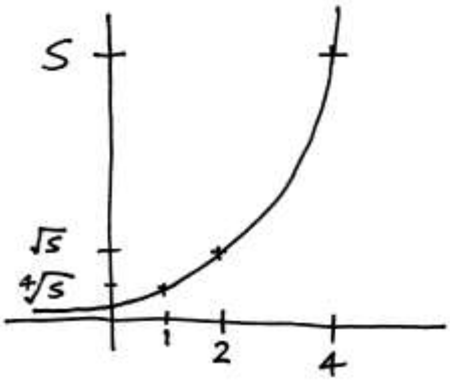
When x is 2 (i.e. at the square root of 4.810477380965^{i4}), the point is half the way around the circle:

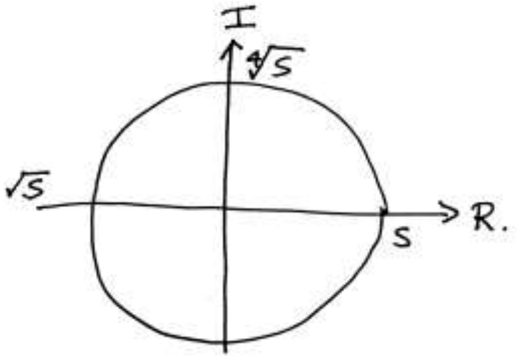


When x is 1 (i.e. at the fourth root of 4.810477380965^{i4}), the point is quarter of the way around the circle:



From this, we could kind of say that the y-axis values on the graph for 4.810477380965^x are being wrapped around the circle, but in a logarithmic way.

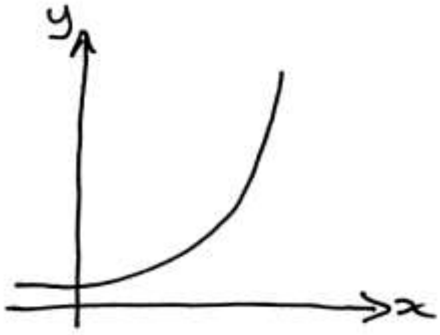




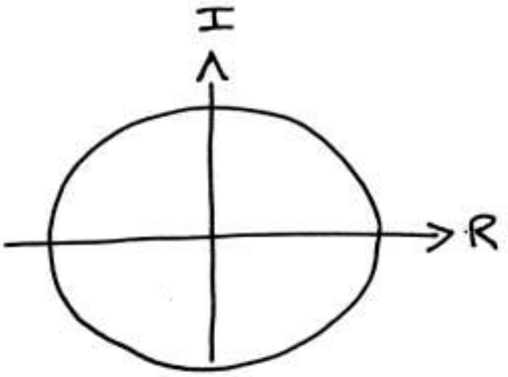
And the same is true for any other base and exponent combination that obeys the formula: b = $\sqrt[c]{S}$

Chapter 7: Increasing Powers of "i"

From what we have learnt, it should now be clear that if we have a Real number to a power of x, it will indicate a point on a curve on x and y-axes:



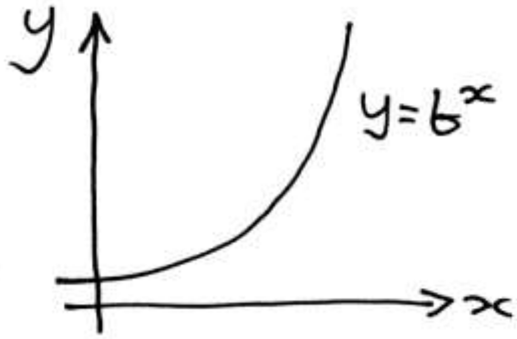
... and if we have a Real number to a power of ix (or in other words, iθ), it will indicate a point on a unit-radius circle on the Complex plane.



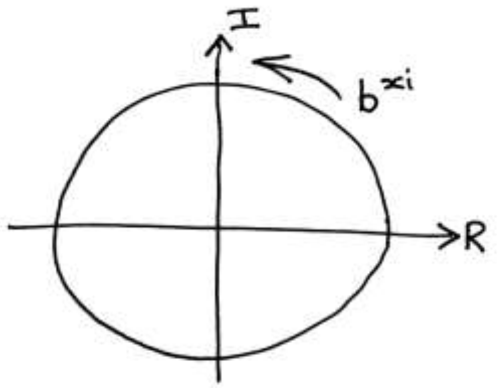
The 4 Stage Cycle of "i" Powers

Where this idea becomes more interesting is when we repeatedly raise an existing exponential to the power of "i". As an example, we will use a generic formula: b^x where "b" is a typical Real number for this sort of thing.

If we were to plot the points "$y = b^x$" over a range of x, we would end up with a curve on x and y-axes like so. As x increases, so does y.

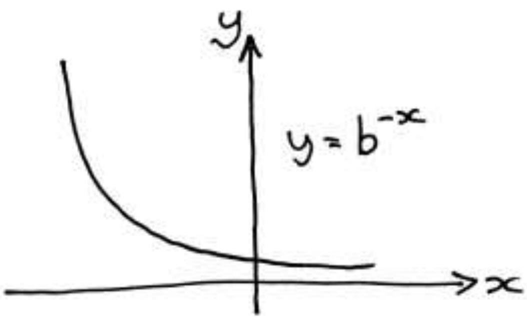


Now, if we raise this entire exponential to the power of "i", we will get: $(b^x)^i = b^{xi}$. In this formula, x is now actually an angle in a system of angles related to the base b. If we were to plot the points of b^{xi} for a range of x, we would get a unit-radius circle. As x increases, b^{xi} indicates a point on the unit-radius circle at ever increasing angles, which is the same thing as saying that as x increases, b^{xi} indicates points that are further anticlockwise around the circle:

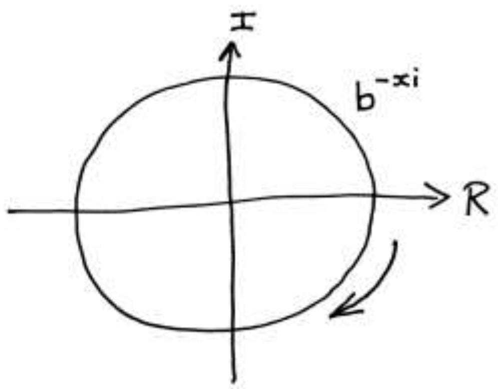


So far, we already knew all of this.

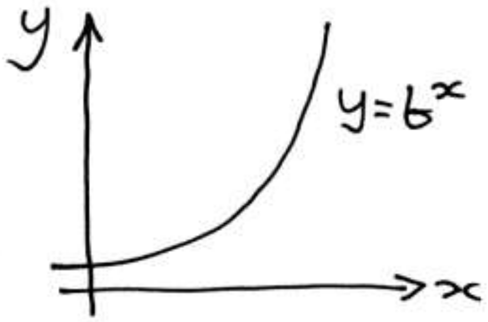
Now, if we raise the entire exponential b^{xi} to the power of "i", we will have $(b^{xi})^i = b^{xii} = b^{x*-1} = b^{-x}$. We have gone from a circle on the Complex plane, back to a curve on x and y-axes. What's more, this time the curve is b^{-x} which is a flipped version of the exponential we started with. As x increases, y decreases. The curve looks like this:



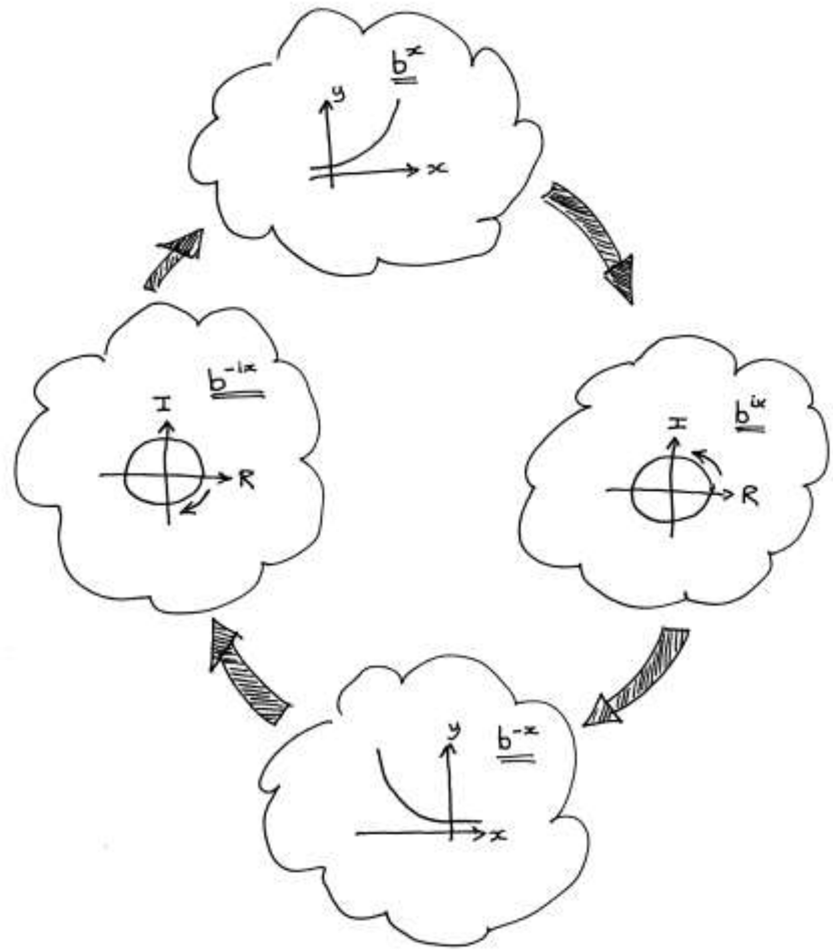
Now, if we raise the entire exponential, b^{-x} to the power of "i", we will have $(b^{-x})^i = b^{-xi}$. We have gone from the backwards exponential curve to a circle on the Complex plane again. But this time, as x increases, b^{-xi} indicates a point on the unit-radius circle at ever *decreasing* angles. In other words, as x increases b^{-xi} indicates points that are further *clockwise* around the circle. This type of circle has parallels with a circle that is divided up into a negative number of portions.



And, if we raise b^{-xi} to the power of "i", we get $(b^{-xi})^i = b^{-xii} = b^{-x * -1} = b^x$. We have got back to the curve on x and y-axes that we started with:



To summarise this series of events, every time we raise something to the power of "i" we switch from a curve on x and y-axes to a circle on the Complex plane, or we switch from a circle on the Complex plane to a curve on x and y-axes. Also, there is a progression, or cycle, of 4 stages that goes: curve, circle, backwards curve, backwards circle. At whichever stage we start, raising the exponential to the power of "i" will move it onto the next stage:



Example With e^x

To show the cycle working in practice, I will show what happens to e^x.

e^x is a curve on x and y-axes. When raised to the power of “i”, it becomes:
$(e^x)^i = e^{xi} = e^{ix}$, which is a circle on the Complex plane.

When that is raised to “i”, we get:
$(e^{ix})^i = e^{xii} = e^{x * -1} = e^{-x}$, which is a “backwards” curve on x and y-axes.

When raised to “i”, we get:
$(e^{-x})^i = e^{-xi} = e^{-ix}$, which is a “backwards” circle on the Complex plane.

When raised to “i”, we get:
$(e^{-ix})^i = e^{-ixi} = e^{-xii} = e^{-x * -1} = e^x$, which is what we started with.

Example With i^x

The formula "i^x" is already referring to a circle on the Complex plane. Therefore, it is at stage 2 in the cycle.

When raised to the power of "i", it becomes:
$(i^x)^i = i^{ix}$, which refers to a backwards curve on x and y-axes.

When that is raised to "i", we get:
$(i^{ix})^i = i^{iix} = i^{-x}$, which refers to a backwards circle on the Complex plane.

When raised to "i", we get:
$(i^{-x})^i = i^{-ix}$, which refers to a forwards curve on x and y-axes.

When that is raised to "i", we get:
$(i^{-ix})^i = i^{-iix} = i^x$, which is what we started with and refers to a forwards circle on the Complex plane.

"i" to the power of "i"

Where the 4 stage cycle becomes useful is in visualising the solving of seemingly difficult calculations such as "i^i" or "$\sqrt[i]{i}$". There are already methods to calculate such things using the fact that "$b^{i\theta} = \cos\theta + i\sin\theta$", but they take some thought. We can visualise what we are doing more easily using the knowledge of the 4 stage cycle.

We know that "i^x" represents a circle on the Complex plane. Therefore, if we raised that to the power of "i", we know that the resulting formula, "i^{ix}", would be the next stage in the 4 stage cycle. Therefore, it would be a backwards curve on x and y-axes. In that case, any results would be nice Real numbers. In other words, for any Real value of x, "i^{ix}" would be a Real number. Therefore, if x is 1, we would have "i^i", and we would know that the solution would be a Real number.

To actually calculate the value of "i^i" is easy. We know that "i^x" divides a circle up into 4 pieces. Using what we know from earlier, this means that it is identical to "4.8105^{ix}". Therefore, "i^{ix}" is identical to "4.8105^{iix}" = "4.8105^{-x}". This confirms that "i^{ix}" is a backwards curve on x and y-axes.

The value "i^i" is equal to "i^{1i}", which means it is the same as 4.8105^{-1}, which is 1 ÷ 4.8105 = 0.207879576351 [to 12 decimal places, and calculated from the full value of 4.810477380965...]

Usually in explanations of calculating "i^i", people are surprised that "i^i" should result in a Real number. This is because they don't realise that raising something to the power of "i" changes the nature of what the exponential is about. Previously, one might have tried to plot the result of "i^i" on the Complex plane – i.e. by marking its position at 0.2079 + 0i, and in doing that, it seems very confusing that something should end up rotated and scaled to that point on the Real axis. However, as we now know, the value 0.2079 in this case doesn't belong on the Complex plane – it belongs on the backwards curve on x and y-axes. It represents the point on the curve of "4.8105^{-x}" where x is 1.

i^{th} Roots of i

An i^{th} root of "i" such as "$\sqrt[i]{i}$" seems completely incomprehensible, but it is actually reasonably easy to solve. [We actually found out what "$\sqrt[i]{i}$" is when we found the base for a circle with 4 divisions in Chapter 6, but we didn't really *solve* it – instead we just knew what its equivalence was].

We'll solve a generic root: the ix^{th} root of i: "$\sqrt[ix]{i}$". We can rephrase this as: $i^{(1\,/\,ix)}$.

This is "i" raised to a multiple of "i". From our knowledge of the 4 stage cycle, we know that "i" raised to a multiple of "i" will have a result that is on the backwards curve on the x and y-axes. Therefore, the result will be a Real number.

We know that "i^x" = "4.8105^{ix}".
Therefore, "$i^{(1/\,ix)}$" will be $4.8105^{(i\,*\,1/ix)} = 4.8105^{(i/ix)} = 4.8105^{(1/x)}$.

Therefore, $\sqrt[i]{i} = \sqrt[1i]{i} = i^{(1/1i)} = 4.8105^{(1/1)} = 4.8105$.

As another example, if we wanted to calculate "$\sqrt[3i]{i}$", it would be equivalent to "$i^{(1/3i)}$" = $4.8105^{(1/3)}$ = 1.688091794964 [to 12 decimal places, and calculated from the full value of 4.810477380965...]

Chapter 8: Some Thoughts

The Problem With "S"

There is one problem in this explanation: we only have one number that we can use to test the validity of S. If we didn't have a way of calculating "e" raised to a power of "i", then we wouldn't be able to know what S is. We are calculating S using "$e^{2\pi}$", and everything else is derived from that. The values of other bases, what i^i is, what the i^{th} root of a number is etc, all depend on how we calculate S. If we said that S were a different number, everything else would have a consistent manner of calculation, but the results would be different. However, if we didn't know about "$e^{2\pi}$", we wouldn't be able to say that the results were wrong because, as far as I can tell, there is no other way of knowing that yet. The values of everything after we rebased "i^θ" with a Real base rely solely on "$e^{2\pi}$". Therefore, ideally, we need ways of calculating S that don't require knowledge of "e".

The True Meaning of $e^{i\theta}$

As we know, using "i^θ" to identify the position of a point on a unit-radius circle's edge is really a method of showing the amount of rotation required to get "1 + 0i" to that particular position. A multiplication by "i^θ", where θ is an angle in a system that divides a circle into 4 parts, results in a point being rotated by an angle of θ. The exponential "i^θ" can be used in two ways: it can be used for rotating a point around the Complex plane, or it can be used to identify the position of a point (by using its rotating properties).

If we, for example, multiply "i^2" by the number 3, we could say two things:
1: We could say we are scaling the unit circle by 3, and so the point on the unit circle at the equivalent of 180 degrees is now at the same angle, but 3 times further away from the origin.
2: We could say that we are rotating the point "3 + 0i" by 180 degrees by multiplying it by "i^2".

These both amount to the same thing. The result is a point at "-3 + 0i". The first way of thinking is possibly the easiest to visualise, but the second is, in my view, a better way of understanding the properties of "i^θ". It reinforces the rotation aspect of a multiplication by "i^θ".

The rotation idea makes things more intuitive when the maths becomes more complicated. For example, if we multiply "3.4 - 7.8i" by "i^3", we know that we are really rotating that point by the equivalent of 270 degrees. The result will be exactly the same distance from the origin.

Some Deductions

We know that multiplying a Complex number by "i^θ" results in a rotation of the point indicated by that number by the angle of θ, where θ is an angle in a system where circles are divided into 4 parts.

We know that the formula of "i^θ" can be rephrased so that the θ can be used to refer to angles in other angle systems. Each resulting formula says an identical thing, but we adjust the size of θ by multiplying it, or dividing it, by a value so we can use other angle systems.

Therefore, multiplying the position of a point by one of these rephrased formulas will still result in a rotation of the point by the angle of θ, where θ is an angle in the relevant system for that formula.

We know that "$i^{(2\theta/\pi)}$" is a rephrasing of "i^θ", where the θ in "$i^{(2\theta/\pi)}$" is an angle in a system where a circle is divided up into 2π divisions. In other words, θ is a number of radians.

Therefore, multiplying a point by "$i^{(2\theta/\pi)}$" will result in the point being rotated by θ number of radians.

We know that "$(\sqrt[i\pi]{-1})^{i\theta}$" is a rephrasing of "$i^{(2\theta/\pi)}$", but with a base to an Imaginary power, instead of an Imaginary base to a Real power.

Therefore, multiplying a point by *that* will result in the point being rotated by θ radians.

We know that "$e^{i\theta}$" is a rephrasing "$(\sqrt[i\pi]{-1})^{i\theta}$", but with a Real base, instead of a base that has an i^{th} root of -1 in it.

Therefore, after all that, we can make the statement:
Multiplying a Complex number by "$e^{i\theta}$" rotates the point indicated by that number by θ radians.

And from this, we can conclude that:

Multiplying any number by "$e^{i\theta}$" is equivalent to treating that number as if it were a Complex number identifying a point, and then rotating it by θ radians.
Such deductions might be blatantly obvious, or they might be slightly profound. However, the basic idea from all this is: We used to treat "i^{θ}" as merely something that *rotates* a point, and now we can use it to *identify* points. Conversely, we used to treat "$e^{i\theta}$" as something that *identifies* a point, but really we should be treating it, primarily, as something that *rotates* a point. It just so happens that anything that rotates a point can be used to identify a point too. The main concept here is that "$e^{i\theta}$" is in essence a means of rotation, and we use that rotation to identify the position of points.

Because we are so used to seeing "$e^{i\theta}$" as a means of identification, it might be difficult to appreciate that it is no different to "i^{θ}" in how it can be used for rotation.

Now we know that if we, for example, multiply a Complex number such as "5.5 - 7.8i" by "e^{2i}", we are really rotating it by 2 radians. The result will be the same distance from the origin.

Similarly, if we, for example, multiply a number such as 10 by "e^{3i}", we are really multiplying the number "10 + 0i" by "e^{3i}", which is the same as rotating the point described by "10 + 0i" by 3 radians. The result, again, will be the same distance from the origin (which in this case is 10 units).

If we multiplied the number "2" by "e^{1i}", 2π times in a row, we would end up with the number 2 – each step rotates it one more radian onwards. [We can also know this is true because it would be "$2 * (e^{i})^{2\pi}$", which is "$2 * e^{2\pi i}$" = 2 * 1 = 2].

What "$e^{i\theta}$" means

After all that, we can say that multiplication by "$e^{i\theta}$" is a method of rotating a point on the Complex plane by θ radians at a time. It should be thought of as being the exact same concept as "i^{θ}" (but obviously rotating a different amount). And, as with "i^{θ}", "$e^{i\theta}$" can also be used to identify the position of a point on a unit-radius circle on the Complex plane, by showing how much rotation is required to get "1 + 0i" to that particular point.

We can also say something similar about all of the exponentials in this explanation. Every exponential is ultimately a means of rotating a point by an angle of θ, where θ is an angle in that particular system. The exponentials can also be used to *indicate* the position of a point by using their rotating properties.

Another thing is that when we say "$e^{i\theta} = \cos\theta + i \sin\theta$", if we were really explaining what this means, it should be:
"$(1 + 0i) * e^{i\theta} = \cos\theta + i \sin\theta$"

...or:
"1 + 0i", when multiplied by "$e^{i\theta}$" results in the point at "$\cos\theta + i \sin\theta$", when θ is in radians.

...or to put it another way:
"1 + 0i", when rotated by θ radians, results in the point at "$\cos\theta + i \sin\theta$".

And, if we look at the equation: "$e^{i\pi} = -1$", it should be:
"$(1 + 0i) * e^{i\pi} = -1 + 0i$"

... or:
"1 + 0i", when multiplied by "$e^{i\pi}$", results in the point at "-1 + 0i".

... or to put it another way:
"1 + 0i", when rotated by π radians, results in the point at "-1 + 0i"

Simpler Maths

The rotation idea of "$e^{i\theta}$" and other related exponentials makes visualising some calculations a lot easier.

If we want to rotate the point "2.2 + 4.7i" by say 3 radians, we can now understand how this would do the job:
$(2.2 + 4.7i) * e^{i3}$.

If we want to rotate the point "1.87 – 9.76i" by 22.5 degrees, we could use:
$(1.87 - 9.76i) * 1.017606491206^{22.5i}$

If we want to rotate the point "6 + 6i" by 0.23 of a circle, then we could use:
$(6 + 6i) * 535.491655524765^{0.23i}$

An Idea For Solving S In a Different Way

Knowing that "$e^{i\theta}$" and all the other exponentials are primarily a means of rotation, suggests to me that we could use that idea to calculate bases without using knowledge of "$e^{2\pi}$", although, I haven't thought of a way to do this yet.

Chapter 9: Waves In Relation to These Circles

Given that a basic Sine wave is made up of the y-axis values of evenly spaced points around a unit-radius circle's edge plotted against their angle, and that a basic Cosine wave is made up of the x-axis values of evenly spaced points around a unit-radius circle's edge plotted against their angle, it should be apparent that "i^{θ}", and every possible circle system derived from it, can be used to describe and analyse waves.

Sine Waves

The standard time-related Sine wave formula using radians is: A sin (2πft + ϕ), where A is amplitude, f is frequency in cycles per second, ϕ is phase in radians, and Sine is operating in radians.

The equivalent for degrees is: A sin (360ft + ϕ), where ϕ is the phase in degrees, and Sine is operating in degrees.

The equivalent for a system that divides a circle up into 4 pieces is: A sin (4ft + ϕ).

The equivalent for a system with a circle in one piece is: A sin(ft + ϕ).
Note how the angular frequency is the same as the cycles-per-second frequency. Therefore, the formula is simpler.

Exponentials

The standard time-related exponential that represents both Cosine and Sine at the same time (i.e. an object moving around a circle) in radians is:
$Ae^{(i2\pi ft + i\phi)}$

The equivalent for degrees is:
$A * 1.0176^{(i360ft + i\phi)}$

The equivalent for a 4-piece circle system is:
$A * 4.8105^{(i4ft + i\phi)}$

The equivalent for a 1-piece circle system is:
$A * 535.4917^{(ift + i\phi)}$.

If we go back to our original system of using "i" raised to a power, we have:
$Ai^{(4ft + \phi)}$

Chapter 10: Conclusion

What Have We Learnt?

From everything in this explanation, we now know:

We can identify a point around a unit-radius circle using just powers of "i". For example, $i^{0.5}$ represents a point on the unit circle at the equivalent of 45 degrees from the origin.

If we do this, there is an implied circle division of 4 portions.

If we do this, we can also say that that point is at "cos θ + i sin θ", where θ is an angle in a system that divides the circle into 4 angles, and Cosine and Sine are operating within this same system.

We can "scale" this exponential to identify a point anywhere on the Complex plane. For example, $2^{i\theta}$ indicates a point 2 units away from the origin at an angle of θ, where θ is an angle in a system based on a circle being divided into 4 parts. Strictly speaking, we are not scaling the exponential, but just rotating a different number by multiplying it against the exponential.

We can adjust this method so θ can be a value representing an angle in a different system of dividing up a circle. For example, if we want θ to be in degrees, we can use $i^{\theta/90}$. Any point referred to in this way can also be referred to using "cos θ + i sin θ", where θ is in degrees, and Cosine and Sine are working in degrees (but only degrees).

As another example, if we want θ to be in radians, we can use $i^{2\theta/\pi}$. Any point referred to in this way can also be referred to using "cos θ + i sin θ", where θ is in radians, and Cosine and Sine are working in radians (but only radians).

We can rephrase the i^{θ} exponential and the various exponentials that came from it, in the form of a base raised to an exponent that is a multiple of "i". These bases will usually mention an i^{th} root of -1 in some form.

Using the knowledge from any known method of raising a value to an Imaginary power for which we know the result, we can convert the untidy bases from the rephrased exponentials into Real numbers. [I only know two methods of raising a value to an Imaginary power that have a known result – Taylor series and the idea of compound interest – and both of those involve raising "e" to "i". There must be others though].

Given that all of these exponentials are ultimately derived from "i^θ", and it is the case that "$i^\theta = \cos\theta + i\sin\theta$", when θ is an angle in a system that divides a circle into 4 portions, and Cosine and Sine are operating in that system, then this relationship still holds – all of the bases when raised to the power of iθ will be equal to "$\cos\theta + i\sin\theta$", when θ is an angle in that particular system of dividing up a circle, and Cosine and Sine are operating in that system.

All the exponentials, including "$e^{i\theta}$", are primarily forms of rotation, however, in the case of "$e^{i\theta}$", we tend not to think of it in this way. An exponential written as "$b^{i\theta}$" should really be thought of as "1 + 0i" multiplied by "$b^{i\theta}$".

A multiplication by an exponential, "$b^{i\theta}$", rotates a point on the Complex plane by the angle θ, where θ is an angle in a system of angles implied by the base.

A multiplication by "$e^{i\theta}$" rotates a point on the Complex plane by θ radians.

A multiplication by "e^{i}" rotates a point on the Complex plane by 1 radian.

Repeatedly raising an exponential to the power of "i" changes the realm to which the exponential applies, by switching realms through a 4-stage cycle. The stages are: curve, circle, backwards curve, backwards circle. Knowing this allows us to know whether a result of an exponential will definitely be a Real number or not. There are probably further uses to this knowledge.

We can see that saying "$e^{i\theta} = \cos\theta + i\sin\theta$" in any angle system except for radians is obviously wrong. The value "e" can only used as a base in system that divides a circle up into 2π portions.

We can use any of these exponentials to represent waves and objects rotating around circles, and there is no particular advantage (that I can see) in using $e^{i2\pi ft}$.

We could, if we wish, use an angle system based on a circle in one piece, and give angles as decimal fractions, which would make a lot of maths a lot more straightforward. Given that Sine and Cosine essentially operate on the fraction of a circle represented by an angle, this makes the most sense to me.

What This Means for the Wonder of "$e^{i\theta}$".

I will comment on the properties of "$e^{i\theta}$" that seem to impress or confuse people the most, and how significant these properties are, now we know what we know.

"$e^{i\theta}$" shows that there is a connection between "e" and "π".

This is still a very interesting fact. There might be other interesting numbers that are also connected.

"$e^{i\theta}$" gives validity to radians.

If you are used to using radians, and then one day you find out that "$e^{i\theta}$" works in radians, it is mind-blowing. It makes the idea of dividing a circle up into 2π divisions seem like the natural way of dividing up a circle. However, from this entire explanation, hopefully, it is apparent that it is difficult to say that one particular way of dividing up a circle is any more natural than another. The breakthrough into circles on the Complex plane usually comes from solving "e^{i}" and not through solving other exponentials, which gives radians more publicity than they might receive otherwise. Of course, there is more to the use of radians than just $e^{i\theta}$ – an angle in radians is the same as the length of the arc of the circumference described by that angle. However, if we are choosing angle systems for the purposes of exponentials, if there is any "most natural" way of dividing up a circle, then it should be into quarters, as happens when using "i^{θ}". The most *useful* way, in my opinion, is keeping the circle as one piece and using decimal fractions.

"$e^{i\theta}$" identifies the position of a point on a unit-radius circle.

The same is true for countless other exponentials. However, we shouldn't really be saying that "$e^{i\theta}$" identifies a position of a point – instead we should be saying that *multiplication* by "$e^{i\theta}$" *rotates* a point by θ radians, and therefore, "1 + 0i" multiplied by "$e^{i\theta}$" allows us to specify the position of a point on the unit-radius circle.

"$e^{i\theta}$" equals "cos θ + i sin θ" when dealing in radians.

After reading this explanation, this equivalence should stop being significant. If someone now says, "$e^{i\theta}$ equals cos θ + i sin θ", the response should be, "Well, yes. It would do."

A more significant idea about the equation is that it should really be:

"1 + 0i" multiplied by "$e^{i\theta}$" results in a point at "cos θ + i sin θ".
or:
"(1 + 0i) * $e^{i\theta}$ = cos θ + i sin θ"

By missing out the "1 + 0i", we lose the sense that "$e^{i\theta}$" is being used as a means of rotation. Of course, the formulas have exactly the same result, and it is customary to ignore "0i", and it is customary not to mention multiplications by 1, but I still think the true sense of the equation is lost without mentioning "1 + 0i" in full.

You can use "$e^{i\theta}$" to describe waves.

We can use countless other exponentials to describe waves too. The simplest being "i^{θ}".

"$e^{i\pi}$ = -1"

I would say that there are 4 different responses that people have to this equation, depending on their level of understanding:

1: They don't understand what it means. No one can be blamed for not understanding the formula as it is only relevant in particular fields.

2: They don't *really* understand what it means, and are amazed by this formula.

3: They understand what it means, but treat its significance in being *what* it says, so aren't that amazed by it. In other words, they see "$e^{i\pi}$ = -1 + 0i" as just saying that a point at the equivalent of 180 degrees on a unit-radius circle is situated at the coordinates (-1, 0). Given that "$e^{i\theta}$" is "cos θ + i sin θ" when θ is in radians, the position of that point is to be expected, so it doesn't really tell us anything we didn't know already.

4: They understand what it means, and treat its significance in being the fact that it is possible to say what it says *in this way*, and therefore they *do* find it interesting. In this sense, the interesting part of "$e^{i\pi}$ = -1" is the fact that without how exponentials and circles on the Complex plane relate to each other, the formula wouldn't make any sense. It *is* interesting how $e^{i\theta}$, or any base raised to iθ, or even i^{θ}, identifies a point at a particular angle on a unit-radius circle on the Complex plane. It is also very interesting how "e" and "π" are related. In the same way that the owner of the world's best fountain pen might write some text to show off the abilities of the pen, so is the formula "$e^{i\pi}$ = -1" really a demonstration of the interesting underlying maths. In that sense, the formula is interesting, and is still interesting even after this entire explanation. It is still a good, succinct way of saying everything that is possible. However, I do think that explanations should point out that it is actually short for: "(1 + 0i) * $e^{i\pi}$ = -1 + 0i".

In my attempt to come up with something similar for the more interesting aspects of everything here, I came up with this slightly clumsy pair of separate but related equations instead:
"$x^x = e^{2\pi}$" and "$x^{xi} = i^4$"
... with the implication that the "x" in both must be 4.304530324517.

You could rephrase the second equation as a riddle: "If x is greater than 1, and $x^{xi} = i^4$, then what is x?"

No equation I could think of seems to say so much in so few symbols as "$e^{i\pi} = -1$". Therefore, in my opinion, it is still the best equation.

Future Research

I think it would be interesting if anyone can come up with a method to calculate the Real number base for a system where a circle is divided into a certain number of portions, without resorting to knowing about "e^i". The gap in the steps from "i^θ" to "$b^{i\theta}$" needs filling, but it is above my level of maths.

Something related to that is finding a way of testing if a number is a valid base for a particular method of dividing up a circle, and doing it without knowledge of "e". There probably is one, but I haven't been able to figure it out yet.

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