Removing the mysteries of "e"
by Tim Warriner

[I originally wrote this in January 2019. My maths and writing skills have improved since then, so I rewrote parts of it in September 2021. If you find this explanation too complicated, you might want to read my book about waves. It starts with explanations of Sine and Cosine and progresses from there. It is available as a free download on my website: www.timwarriner.com]

Introduction

Two of the equations that get the most interest in the world of maths are:
"$e^{ix} = \cos x + i \sin x$"
... and the related:
"$e^{i\pi} = -1$"

In all the explanations that I have seen about "e^{ix}", there is always an unexplained gap between the meaning of the exponential "e^x" and the meaning of the Complex exponential "e^{ix}". There is never any sensible reason given as to why an exponential curve should end up as a circle on the Complex plane. In most school education, we are just told to accept it all without question. It is hard to visualise why "e^{ix}" does what it does.

In this explanation, I will try to remove the obscurity of the meaning of "e^{ix}" and its related formulas by demonstrating how it works.

To get the most from the following explanation, it will help if you have at least a trivial understanding of Complex numbers and exponentials, and it might help if you have some experience of "e^{ix}". A good test for whether you will get what I am trying to say or not is whether you understand how Sine and Cosine indicate the y-axis and x-axis values of points on the circumference of a unit-radius circle at particular angles from the centre. It is possible to calculate the Sine and Cosine of a number by drawing and measuring a circle. If you understand this, then this explanation should be straightforward. If you do not understand this, then you might benefit from my book about waves, which is available on my website at:
www.timwarriner.com

Notes

- Given that "x" in "e^{ix}" is actually an angle, in this explanation, I will write the exponential as "$e^{i\theta}$" to be consistent with using "θ" for angles. For example, instead of "$e^{ix} = \cos x + i \sin x$", I will write "$e^{i\theta} = \cos \theta + i \sin \theta$".

- This explanation is meant to be easy to understand. Therefore, there are things that will be obvious to people who are good at maths, and also things that might be obvious to everyone. It might seem overly repetitive at times.

- I frequently include unnecessary ones and zeroes in Complex numbers in order to make the explanation clearer.

- I tend to capitalise certain words to make ideas clearer and less ambiguous. For example, I refer to "Complex" numbers rather than "complex" numbers.

- For decimal fractions, I tend to round the value to 4 decimal places unless the number is particularly significant at that moment.

Summary

The following chapters will be a simple exploration of the following important ideas:

- It is possible to identify any point on a unit-radius circle by saying by how much the point at "1 + 0i" would need to be rotated to get there.

- It is possible to rotate a point on the Complex plane by 90 degrees (0.5π radians) by multiplying the point by "i".

- It is possible to rotate a point on the Complex plane by a particular angle in an angle system that divides a circle into four pieces, by multiplying the point by a power of "i" where the exponent is that angle. In other words, if we multiply a point on the Complex plane by "i^{θ}", where "θ" is an angle in an angle system based on a circle being divided into four pieces, that point will be rotated by that angle. We will refer to angles in a system where a circle is divided into four pieces as "quarter-circle angle units".

- We can identify a point on a unit-radius circle by saying by how much the point at "1 + 0i" would need to be multiplied by "i^{θ}" to get there. That point will be 1 unit away from the origin and at an angle of "θ" quarter-circle angle units.

- Therefore, we can identify any point on a unit-radius circle solely in terms of "i^{θ}".

- Therefore, we can say that any point indicated by "i^{θ}" can also be indicated by "cos θ + i sin θ", where Cosine and Sine are working in quarter-circle angle units. As the point indicated by "i^{θ}" is one unit away from the origin and at an angle of "θ" quarter-circle angle units, then it must be the case that it is equal to "cos θ + i sin θ", which identifies the same point when Cosine and Sine are working in quarter-circle angle units.

- We can identify *any point on the Complex plane* by saying by how much the point at "1 + 0i" must be rotated and scaled to get there in terms of *multiples* of "i^{θ}". This means that "ai^{θ}" identifies any point on the Complex plane, where "a" is the distance from the origin, and "θ" is an angle in quarter-circle angle units.

- We can rephrase "i^{θ}" so that "θ" can be a value in degrees or radians or any other way of dividing up a circle. For example, "$i^{(\theta/90)}$" works with "θ" as an angle in degrees. If we enter an angle in degrees into "$i^{(\theta/90)}$", the result will be a point on a unit radius circle at an angle of "θ" degrees. Therefore, the exponential "$i^{(\theta/90)}$" must be equal to "cos θ + i sin θ" when "θ" is an angle in degrees, and Cosine and Sine are working in degrees. The exponential "$i^{(2\theta/\pi)}$" works with "θ" in radians. If we enter an angle in radians, it will identify a point on a unit radius circle at angle of "θ" radians. Given that, "$i^{(2\theta/\pi)}$" must be equal to "cos θ + i sin θ" when "θ" is an angle in radians, and Cosine and Sine are working in radians.

- We can rephrase the different versions of the exponentials to be a Real base raised to an Imaginary power, and keep the meanings exactly the same. For example, "$(\sqrt[90i]{i}\ )^{i\theta}$" works in degrees; "$(\sqrt[0.5i\pi]{i}\ )^{i\theta}$" works in radians. We might not be able to solve these with the knowledge we have so far in this explanation, but we can prove that they are correct. It is still the case that these are equal to "cos θ + i sin θ", where "θ", Cosine and Sine are working in that particular way of dividing up a circle.

- From knowing that "$e^{i\theta}$" also identifies a point on a unit radius circle at an angle of "θ" radians, we know that "($\sqrt[0.5i\pi]{i}$)" must be equal to "e". From that, we have the step from Real powers of "i" to Imaginary powers of "e".

- Therefore, we can think of "$e^{i\theta}$" as really being a way of identifying the position of a point on a unit-radius circle in terms of how much the point at "1 + 0i" would need to be rotated to get there. The amount of rotation is given by a multiplication by "$e^{i\theta}$".

- We can use what we have learnt to calculate other bases to Imaginary powers. For example, "$1.01761^{i\theta}$" identifies a point on a unit-radius circle at an angle of "θ" degrees. Therefore, "$1.01761^{i\theta}$" must be equal to "cos θ + i sin θ", where Cosine and Sine are working in degrees.

- We can use what we have learnt to solve seemingly difficult calculations such as the i^{th} root of "i".

All of the above ideas mean that the properties of "$e^{i\theta}$" are not mysterious or complicated. They are also not unique. The equivalence of "$e^{i\theta}$" to "cos θ + i sin θ" is something that should be expected, and not a surprising revelation.

There are ideas here that I have not seen elsewhere – I am fairly sure that most people do not know anything about them. Even if the ideas here are not particularly profound, they will help some people better understand some aspects of maths.

Chapter 1: Background

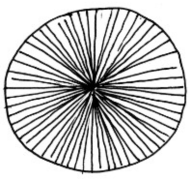
This chapter is designed to set out the basis of the ideas in this explanation, and to get everyone thinking in the same way.

Ways of dividing up a circle

An angle is a value that represents the steepness of a line in terms of a division of a circle. The steepness of one angle unit depends on the number of pieces into which a circle has been divided to create that angle system. There are countless ways to divide up a circle, and, therefore, there are countless angle systems.

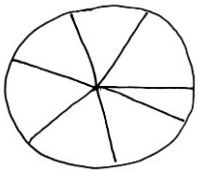
360 pieces

We can base an angle system on a circle divided into 360 pieces. Each angle unit will differ in steepness by an amount equal to one division of such a circle. Each angle will represent one 360^{th} of a circle. As we know, such angles are called "degrees". In this system, 90 angle units represent a quarter of a circle, 180 angle units represent half a circle, and 360 angle units represent a whole circle.



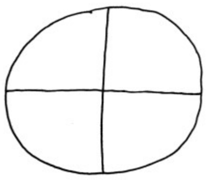
2π pieces

We can base an angle system on a circle that has been divided into 2π pieces (6.2831... pieces). Each angle will be one $2\pi^{th}$ of a circle. As we know, we call such angles "radians". In this system, 0.5π angle units represent a quarter of a circle, π angle units represent half a circle, 2π angle units represent a full circle.



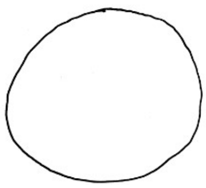
4 pieces

We could also base an angle system on a circle that has been divided into 4 pieces. In such a system, each angle will be a quarter of the circle. One angle unit represents a quarter of a circle, 2 units represent half a circle, and 4 units represent a full circle. In this explanation, we will call these angles "quarter-circle angle units".



1 piece

We could also base an angle system on a circle that remains in 1 piece. If we do this, one unit represents the full circle, half a unit represents half a circle, and quarter of a unit represents quarter of a circle. In this explanation, we will call these angles "whole-circle angle units".



Other Systems

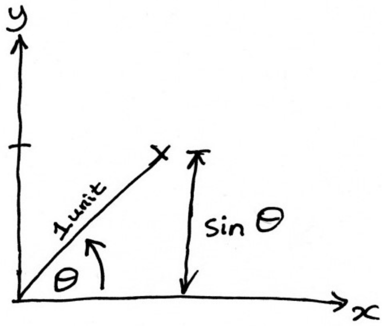
There are countless other ways to divide up a circle, but in this explanation, we will focus on these four angle systems.

Conversion

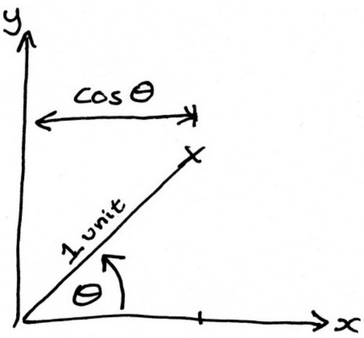
To convert a value from one angle system to another, it is first necessary to find out what fraction of a circle that value represents. We then multiply that fraction by the number of divisions in the system into which we are converting. As an example, we will convert 217 degrees into radians. 217 degrees is 217 ÷ 360 = 0.6028 of a circle. That fraction of a circle is 0.6028 * 2π = 3.7874 radians.

Sine and Cosine

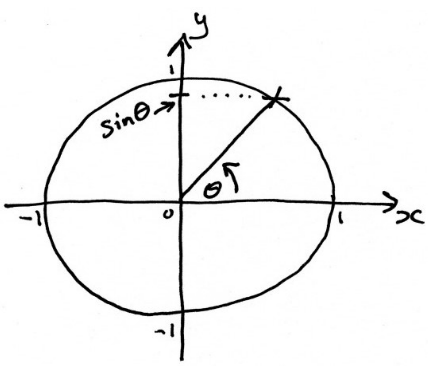
If we consider points on x and y axes, the Sine of a value gives the y-axis value of a point that is one unit away from the origin and at the angle of that value:

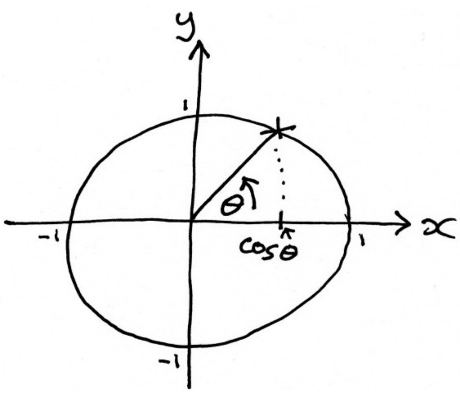


The Cosine of a value gives the x-axis value of a point one unit away from the origin and at the angle of that value:



On a unit-radius circle that is centred on x and y axes, the Sine of the angle of any point on the circumference will be that point's y-axis value. The Cosine of the angle of any point will be that point's x-axis value.



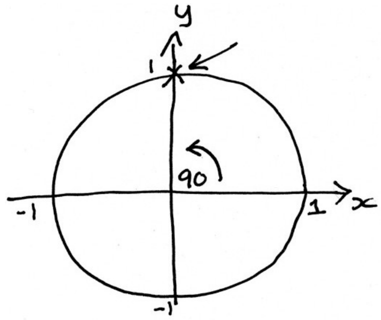


Thinking of this the other way around, any point on a unit-radius circle can be identified by giving its coordinates in terms of the Cosine of the angle and the Sine of the angle. For example, a point at an angle of 30 degrees will have an x-axis coordinate of Cosine 30 and a y-axis coordinate of Sine 30, when Cosine and Sine are working in degrees. Its coordinates will be (cos 30, sin 30).

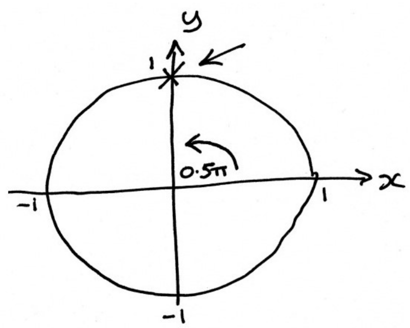
Sine and Cosine and different angle systems

The Sine and Cosine functions give results based on the *portion* of a circle that is represented by an angle. If that portion is being measured in degrees, then the Sine and Cosine functions must be working in degrees to treat that angle as the correct portion of a circle. If that same portion is being measured in radians, then the Sine and Cosine functions must be working in radians to treat that angle as the correct portion of a circle. If that portion is being measured in quarter-circle angle units, then the Sine and Cosine functions must be working within that system too. Given that there are countless ways to divide up a circle, there are also countless ways in which the Sine and Cosine functions can operate to work with these systems. The Sine function on any particular *portion* of a circle will give the same result whether that portion is measured in degrees, radians or any other system of angles. The same is true for the Cosine function.

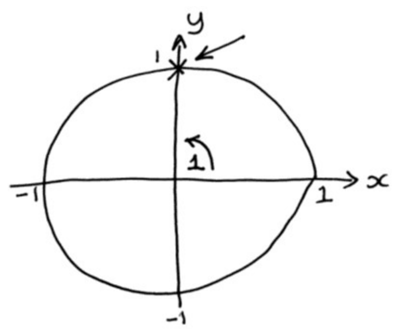
We will now go back to the idea that any point on a unit-radius circle can be identified by giving its coordinates in terms of the Cosine of the angle and the Sine of the angle. If we are measuring angles in degrees, we can indicate a point’s position by giving the coordinates in terms of the Cosine of that angle in degrees and the Sine of that angle in degrees, *when Sine and Cosine are working in degrees*. For example, the point at 90 degrees can be identified using the coordinates (cos 90, sin 90). This results in the coordinates (0, 1).



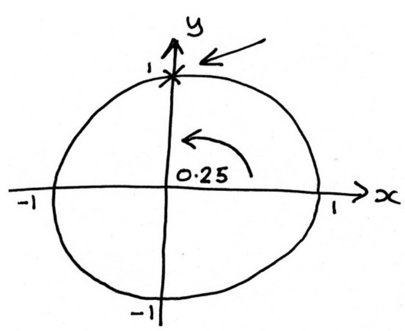
If we were using radians, that same point would be at an angle of 0.5π radians. Its coordinates would be (cos 0.5π, sin 0.5π), *when Sine and Cosine are working in radians*, which also results in (0, 1).



If we were using quarter-circle angle units, that same point would be at an angle of 1 quarter-circle angle unit. Its coordinates would be (cos 1, sin 1), where Cosine and Sine are working in this system of dividing a circle into four parts. The coordinates would still end up as (0, 1).



If we were using whole-circle angle units, that same point would be at an angle of 0.25 whole-circle angle units. Its coordinates would be (cos 0.25, sin 0.25), where Cosine and Sine are also working in this system of the circle being in one part. The coordinates would still end up as (0, 1).



The Complex plane

If we draw a unit-radius circle on the Complex plane, instead of describing points on the circumference in terms of coordinates in the form (x, y), we can use Complex numbers instead. For our point at (0, 1), we would use the Complex number "0 + 1i". [This would normally be written without the unneeded zero and one, but I am keeping them in to make this explanation clearer.]

We can use the Cosine and Sine of an angle in the form of a Complex number to identify the position of our point:

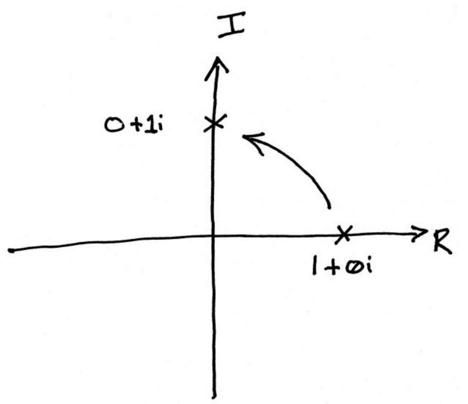
- If we are using degrees, we can say that our point is at "cos 90 + i sin 90".
- If we are using radians, our point is at "cos 0.5π + i sin 0.5π".
- In quarter-circle angle units, our point is at "cos 1 + i sin 1".
- In whole-circle angle units, our point is at "cos 0.25 + i sin 0.25".

These all amount to exactly the same thing.

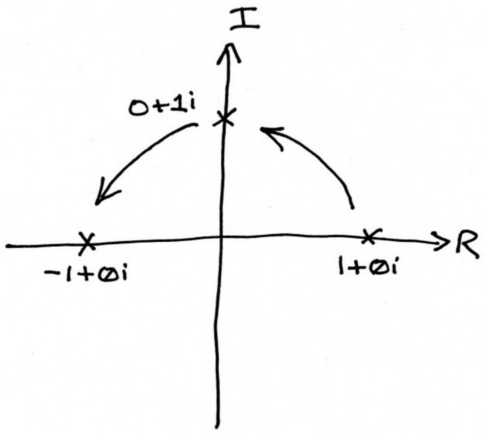
We can identify any point on a unit-radius circle on the Complex plane using this idea. We can also identify any point on the Complex plane at all by scaling the Cosine and Sine calculations. For example, the point at:
"0 + 2i"
... is at:
"2 cos 90 + 2i sin 90"
... when Cosine and Sine are working in degrees.

Multiplication by powers of i

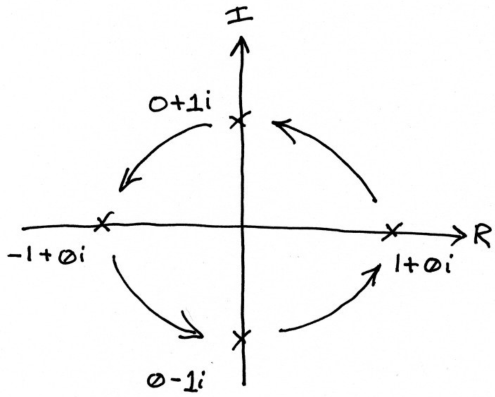
On the Complex plane, a multiplication by "i" is equivalent to a rotation of 90 degrees anticlockwise. Therefore, if we have a point at "1 + 0i", and we multiply it by "i", we end up with a point that is the same distance from the origin, but rotated by 90 degrees. It will be at "0 + 1i":



If we then multiply the resulting point by "i", we end up with a point 90 degrees further around:



If we multiply again by "i", we end up with a point at 270 degrees, and if we multiply by "i" again, we end up with a point at 360 degrees, which is the same as 0 degrees.



From this, we can say that:
A multiplication by "i" is equivalent to a rotation by 90 degrees.
A multiplication by "i^2" is equivalent to a rotation by 180 degrees.
A multiplication by "i^4" is equivalent to a rotation by 360 degrees.

Another way of thinking about this is as so:
Multiplying by $\sqrt{-1}$ rotates by 90 degrees.
Multiplying by −1 rotates by 180 degrees.
Multiplying by +1 rotates by 360 degrees.

Normally, we would say that a multiplication by +1 has no effect, but for consistency's sake, in this case, we will say that it rotates by 360 degrees.

We can also say that a multiplication by "i^0" is equivalent to a rotation by 0 degrees. Technically, "i^0" is the same as "i^4", but by thinking of "i^4" differently, it can help with this whole explanation.

Knowing that we can scale the exponential of "i" to achieve a particular rotation leads us to be able to perform other rotations. For example, a multiplication by "$i^{0.5}$" will rotate a point by 45 degrees. "$i^{0.5}$" is the square root of "i". The square root of *that* ("$i^{0.25}$") will rotate a point by 22.5 degrees.

In each of the above examples, the exponent of "i" is actually the number of quarter-circle angle units by which the point is being rotated. This leads to the idea that we can perform rotations of *any* amount by multiplying by the appropriate power of "i". If we want to rotate a point by a particular angle in quarter-circle angle units, we just put that angle as the exponent, and a multiplication by that power of "i" will rotate a point by that angle. To put this more mathematically, to rotate a point by "θ" quarter-circle angle units, we multiply the point by "i^θ". The exponent dictates the rotational amount. As an example, a multiplication by "$i^{0.123}$" rotates a point by 0.123 quarter-circle angle units.

As it is easier to use degrees or radians than it is to use quarter-circle angle units, we will usually have to convert an angle into quarter-circle angle units to use this idea. To calculate what exponent is needed to rotate by a particular number of degrees, we have to find out what portion of a circle those degrees represent, and then multiply that by 4 to find out how many quarter-circle angle units that is. For example, if we want to rotate a point by 10 degrees, we convert 10 degrees into quarter-circle angle units and use the result as the exponent. The angle of 10 degrees is 10 ÷ 360 = 0.02778 of the way around a circle. This is 0.02778 * 4 = 0.1111 quarter-circle angle units. Therefore, the exponent in our exponential will need to be 0.1111. A multiplication by "$i^{0.1111}$" will rotate a point by 10 degrees.

To rotate a point by 0.12π radians, we first find out what portion of a circle is represented by 0.12π radians. It is 0.12π ÷ 2π = 0.06 of a circle. We then see how many quarter-circle angle units that is by multiplying that by the number of quarter-circle angle units in a circle. This results in 0.06 * 4 = 0.24 quarter-circle angle units. Therefore, to rotate a point by 0.12π radians, we need to multiply it by "$i^{0.24}$".

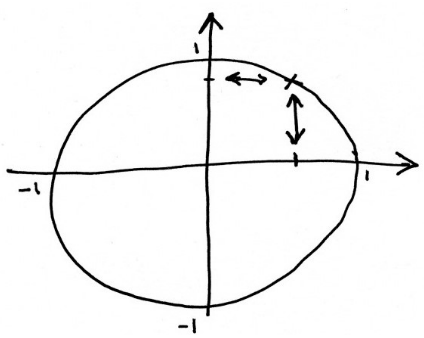
Any form of rotation, no matter how big or small can be achieved by multiplying a point by a power of "i".

Ways of identifying a point on a unit-radius circle

In this section, I will show some methods for identifying the position of points on a unit-radius circle’s circumference. This repeats some of the things that I have already said.

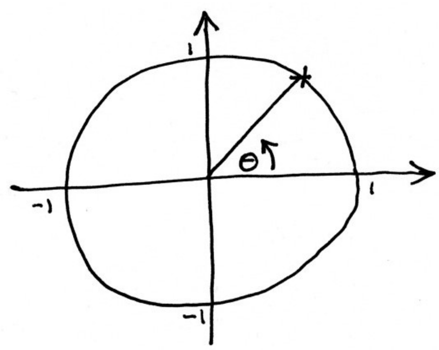
X-axis and y-axis values

We can identify a point on a unit-radius circle by its x-axis and y-axis coordinates. The coordinates of a point on a unit-radius circle can also be thought of as the Cosine and Sine of that point’s angle from the centre of the circle. We can also use Complex numbers in essentially the same way as coordinates.



Angle

If we know that a point is on a unit-radius circle’s circumference, we can indicate its position using just its angle from the origin:



Rotational amount in relation to the point at 0 degrees

We can identify a point on a unit-radius circle by saying by how many angles the point at 0 degrees would need to be rotated to get to that position. In other words, if a point is an eighth of the way around a circle, we could say that the point at 0 degrees would need to be rotated by 45 degrees to get there. Although, some people would say that this is no different to giving its angle (which is true if you only consider the end product), the nuance is different. We are not giving the absolute angle value – instead we are specifically giving the rotational amount. They both amount to exactly the same thing, but the thinking behind them is different.

Chapter 2: i raised to a power

Position in terms of multiplications by "i"

We can use the "rotational amount" idea on the Complex plane. In doing this, we will be identifying a point's position on a unit-radius circle's circumference by saying how much the point at "1 + 0i" would need to be rotated to end up at the position of our point.

Given that the value "1 + 0i" would require a different amount of rotation to get to any particular point on a unit-radius circle's edge, the position of any point can be identified solely by that particular amount of rotation.

As we saw in Chapter 1, we can perform particular amounts of rotation by multiplying a number by "i" raised to an power. The exponent dictates how much rotation happens.

Therefore, we can indicate amounts of rotation using "i" raised to a power.

Therefore, we can *identify* individual points around a unit-radius circle using "i" raised to particular powers.

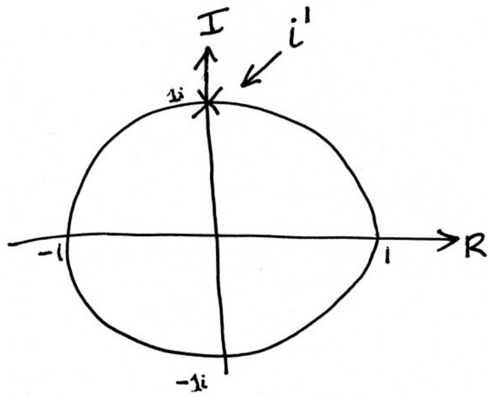
In this sense, we are really multiplying "1 + 0i" by powers of "i", which is the same as multiplying 1 by powers of "i", which is really just the same as showing powers of "i".

The basic rule for this idea, and the basis for this entire explanation is that *any point on a unit-radius circle can be identified using just "i" raised to a power.*

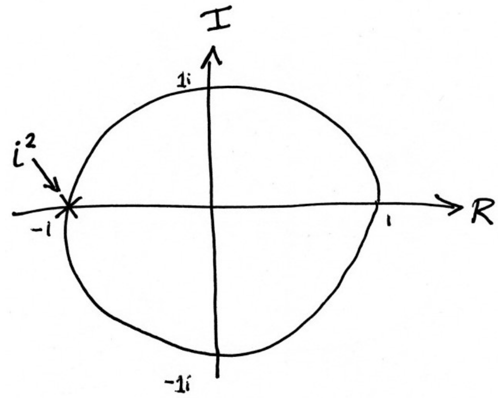
Some examples

A point that is at 90 degrees on a unit-radius circle, or in other words at "0 + 1i", can be identified by the fact that "1 + 0i" would have required a rotation of 90 degrees to get there. In terms of "i", it must have been multiplied by "i^1" in order for it to be there. Therefore, just the exponential "i^1" is sufficient to identify the position of that point.

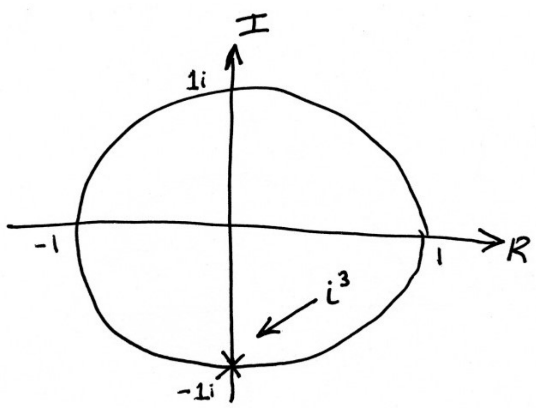
No other point is represented by "i^1", so "i^1" exclusively indicates the position of our point:



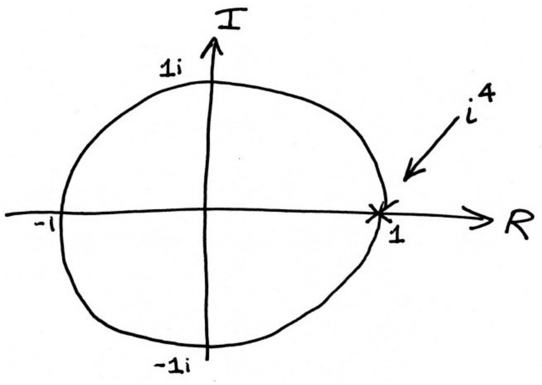
A point at 180 degrees on a unit-radius circle can be identified by "i^2". The point "1 + 0i" would have needed a rotation of 180 degrees to get there. Multiplying by "i^2" is the same as a rotation of 180 degrees.



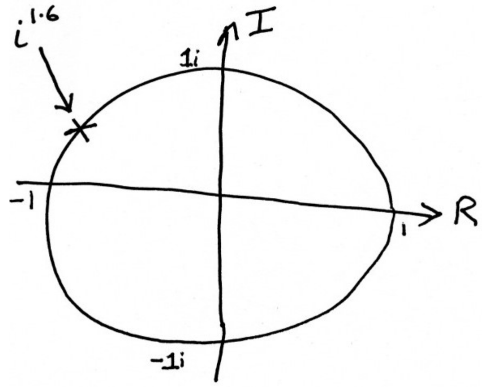
A point at 270 degrees on a unit-radius circle can be identified by "i^3". The point at "1 + 0i" would have required a rotation of 270 degrees to get there, and this is equivalent to a multiplication by "i^3".



A point at 360 degrees on a unit-radius circle can be identified by "i^0" or by "i^4". We could say that the point at "1 + 0i" would have required no rotation to end up where it started, or we could say that it would have required a rotation of 360 degrees to get there.



As a more complicated example, the exponential "$i^{1.6}$" indicates a point that is at "−0.8090 + 0.5878i" [to 4 decimal places]. The point at "1 + 0i" would have required a rotation equivalent to a multiplication by "$i^{1.6}$" to end up at "−0.8090 + 0.5878i".



Remember that in all of these examples, the exponent is actually an angle in quarter-circle angle units.

Calculating “i” raised to a power

Note that we do not need a calculator or knowledge of “$e^{i\theta}$” to calculate “i” raised to any power. We can calculate it in the same way that we can calculate the Sine and Cosine of any angle – by drawing and measuring a unit-radius circle.

We will calculate where the point identified by “$i^{3.4}$” is. [We can think of this point as being the point that is reached by multiplying “1 + 0i” by “$i^{3.4}$”.] First, we draw a large circle on a piece of graph paper with a radius of 1 unit. In practice, it is easiest to draw a circle with a radius of 10 centimetres, and treat 10 centimetres as being 1 unit. Any measurements in centimetres from the circle will need to be divided by 10 to become “units”.

Given that powers of “i” have an exponent that indicates the angle in quarter-circle angle units, we just need to mark the point that is 3.4 quarters of the way around the circle. This is 3.4 ÷ 4 = 0.85 of the way around the circle. Given that protractors generally work in degrees, it is easiest to convert this into degrees and use the protractor to mark the point. 0.85 of the way around the circle is 0.85 * 360 = 306 degrees. Therefore, we draw a line at 306 degrees from the centre of the circle out to its circumference. Where that line crosses the circumference will be the position of the point indicated by “$i^{3.4}$”.

We measure the y-axis position of this point in centimetres, and divide by 10 to convert the measurement into units. When doing this, you should be able to get a result to an accuracy of 2 decimal places, which will be −0.81 units. We then measure the x-axis position of the point, and convert the measurement into units. Again, you should be able to get an accuracy of 2 decimal places. The value will be 0.59 units. The coordinates of the point are (0.59, −0.81). The Complex number that marks this point is therefore, “0.59 – 0.81i”. We can say that:
$i^{3.4} \approx 0.59 - 0.81i$

We can check this with a calculator that can work with Complex numbers. A calculator will give the result of “$i^{3.4}$” as “0.5878 – 0.8090i” to four decimal places. Therefore, our reading from the circle was reasonably accurate.

Given that we know the point is at 306 degrees, we could have skipped drawing a unit-radius circle, and instead just used Sine and Cosine on a calculator to work out the result. The Sine of 306 degrees is −0.8090; the Cosine of 306 degrees is 0.5878. Therefore, the Complex number that indicates the point is “0.5878 – 0.8090i”.

We can make formulas to solve powers of “i” with Sine and Cosine using the fact that the exponent is an angle in quarter-circle angle units. Generally, if we want to convert an angle in quarter-circle angle units to degrees, we first divide the angle by 4 to see how much of a circle that angle represents. We then multiply that by 360 to find out how many degrees that is. Using this knowledge, we can create a formula to solve powers of “i” using Cosine and Sine when they are working in degrees. The formula is:
$i^\theta = \cos(360 * (\theta \div 4)) + i \sin(360 * (\theta \div 4))$
... which we can make more concise as:
$i^\theta = \cos(90 * \theta) + i \sin(90 * \theta)$

Generally, if we want to convert an angle in quarter-circle angle units to radians, we first divide the angle by 4 to see how much of a circle the angle represents. We then multiply that by 2π to find out how many radians that is. Using this knowledge, we can create a formula to solve powers of “i” when Cosine and Sine are working in radians. It is:
$i^\theta = \cos(2\pi * (\theta \div 4)) + i \sin(2\pi * (\theta \div 4))$
... which we can make more concise as:
$i^\theta = \cos(0.5\pi * \theta) + i \sin(0.5\pi * \theta)$

If we had a calculator that could make Sine and Cosine work in quarter-circle angle units, we would just need this formula:
$i^\theta = \cos(\theta) + i \sin(\theta)$

Extending the idea

So far, we know that:

- “i” raised to a Real power identifies any point on a unit-radius circle’s edge.
- The exponent of “i” is really an angle in a system that divides the circle into four pieces.
- We can think of “i” raised to a Real power as being a form of identification, or as a form of identification via a multiplication by “1 + 0i”.

Other radiuses

We can also identify points on circles that have radiuses of lengths other than 1 unit. To do this we, just scale the whole of the “i^θ” value accordingly.

If we want to indicate the position of a point on a circle with a radius of 0.5 units, we can give the position as:
$0.5 * (i^\theta)$
... which is:
$0.5i^\theta$

If we want to indicate a point on a circle with a radius of 2 units, we can give its position as:
$2i^\theta$.

Depending on how we think about the exponential, we could say either of these:

- We are *scaling* the result of “i^θ” by a chosen value.
- We are *rotating* that chosen value (treated as a Complex number at 0 degrees) by the amount dictated by a multiplication by “i^θ”. In this way of thinking, the 2 in “$2i^\theta$” can be thought of as the Complex number “2 + 0i” [the point on a 2-unit-radius circle at 0 degrees] rotated by “i^θ”.

These ways of thinking obviously amount to the same thing, but you might find one easier to visualise than the other.

Given that any point at all in the Complex plane can be thought of as being on the circumference of a circle, we can actually identify *any* point in the Complex plane by using numbers multiplied by exponentials with “i” as the base.

We have come up with a way of identifying any point on the Complex plane using an exponential, but in a way that is straightforward and intuitive. I hope that this explanation is clear enough that anyone with a slightly technical mind can understand this. This idea does not require any huge leaps of faith.

Other types of angles

The value of “θ” in “i^{θ}” is an angle in quarter-circle angle units. To put this more pedantically, the nature of how exponents of “i” work means that “θ” will be treated as if it were an angle in quarter-circle angle units. To make the exponential easier to use, we can adjust it so that we can have “θ” as angles in other angle systems.

Degrees

If we want to use the “i^{θ}” formula with degrees, we need to convert our angle in degrees to become an angle in quarter-circle angle units. To do this, we divide the angle in degrees by 360 to find out what portion of a circle it represents, and then we multiply that by 4 to find out how many quarter-circle angle units that is. When we put this conversion into the exponent of “i”, we end up with this:
“$i^{(4*(\theta/360))}$”
... which becomes:
“$i^{(\theta/90)}$”

This allows us to use the exponential with “θ” as an angle in degrees to identify the position of any point on a unit-radius circle. Note that “θ” is an angle in degrees, but the exponent *as a whole* is in quarter-circle angle units (as any exponent of “i” always will be).

If we want to identify the point at 45 degrees, we can give its position as:
“$i^{45/90}$”
= “$i^{0.5}$”
This is “0.7071 + 0.7071i”, which shows that the idea works.

If we have a point at 252 degrees, we can give its position as:
“$i^{252/90}$”
= “$i^{2.8}$”
This is “−0.3090 – 0.9511i”.

Given that any point on a unit-radius circle at a particular angle in degrees will have an x-axis value equal to the Cosine of that angle, and a y-axis value equal to the Sine of that angle (when Cosine and Sine are working in degrees), we can say:
“$i^{(\theta/90)} = \cos\theta + i\sin\theta$”
... when “θ” is an angle in degrees, and Cosine and Sine are working in degrees.

Radians

If we want to rephrase the "i^θ" formula so that we can use it with "θ" as an angle in radians, we need to convert the radians angle into quarter-circle angle units. To do this, we divide the radians angle by 2π to find out what portion of a circle it represents, and then we multiply that by 4 to see how many quarter-circle angle units it is.

The formula becomes:
"$i^{(4 * (\theta/(2\pi)))}$"
... which is:
"$i^{(2\theta/\pi)}$"
... where "θ" is an angle in radians.

The formula "$i^{(2\theta/\pi)}$" allows us to use the exponential with "θ" as an angle in radians to identify the position of any point on a unit-radius circle. Note that "θ" is an angle in radians, but the exponent *as a whole* is in quarter-circle angle units.

If a point is at 0.25π radians, its position is:
"$i^{(2 * 0.25\pi)/\pi}$"
= "$i^{0.5}$"
This is "0.7071 + 0.7071i".

If a point is at 1.23π radians, its position is:
"$i^{(2 * 1.23\pi)/\pi}$"
= "$i^{2.46}$"
This is "−0.7501 – 0.6613i".

Given that any point on a unit-radius circle at a particular angle in *radians* will have an x-axis value equal to the Cosine of that angle, and a y-axis value equal to the Sine of that angle (when Cosine and Sine are working in radians), we can say:
"$i^{(2\theta/\pi)} = \cos\theta + i \sin\theta$"
... when "θ" is an angle in radians, and Cosine and Sine are working in radians.

Whole-circle angle units

If we want to rephrase the "i^θ" formula so that we can use it with "θ" as an angle in whole-circle angle units, we need to convert the whole-circle angle into quarter-circle angle units. As whole-circle angle units represent a portion of a circle, the conversion just involves multiplying the angle by 4. We can adjust our formula accordingly.

The position of any point on a unit-radius circle can be given as:
“$i^{4\theta}$”
... where “θ” is an angle in whole-circle angle units.

Note that “θ” is an angle in whole-circle angle units, but the exponent *as a whole* is in quarter-circle angle units.

If a point is at 0.25 whole-circle angle units, in which case, it is 0.25 of the way around the circle, its position is:
“$i^{4*0.25}$”
= “i^{1}”
This is “0 + 1i”.

If a point is at 0.5 whole-circle angle units, in which case, it is half way around the circle, its position is:
“$i^{4*0.5}$”
= “i^{2}”
This is “−1 + 0i”.

In this system, “$i^{4\theta} = \cos\theta + i\sin\theta$”, when “θ” is an angle in whole-circle angle units, and Cosine and Sine are operating in whole-circle angle units.

Chapter 3: Changing the bases

So far, the formulas for describing the position of a point around a unit-radius circle are as follows:

- For “θ” as an angle in quarter-circle angle units: “i^{θ}”
- For “θ” as an angle in whole-circle angle units: “$i^{4\theta}$”
- For “θ” as an angle in degrees: “$i^{\theta/90}$”
- For “θ” as an angle in radians: “$i^{2\theta/\pi}$”

We could, of course, have formulas for countless other ways of dividing up a circle.

For each of these, putting “θ” into the formula immediately converts the angle unit into quarter-circle angle units. In other words, “θ” is the angle in a particular type of angle units, but the exponent as a whole is an angle in quarter-circle angle units.

We can change each of these formulas so that instead of being “i” raised to a power, they become a value raised to a multiple of “i”. The actual meaning of the exponentials and what they do will be identical. All we are doing is changing how the calculations are formed.

The basic idea here is that we will convert:
“$i^{\theta?}$” to “$c^{i\theta}$”
...where “θ” represents an angle in a particular system of dividing up a circle, “?” represents some value multiplying or dividing “θ”, and “c” is the new base for the exponential. The value represented by “c” will be a Real number.

Doing this will result in the meaning of the exponential as a whole become more abstract. We will go from something that we can easily visualise, to something that, to start with, will probably seem incomprehensible.

Quarter-circle angle units

We will start with quarter-circle angle units, which use the simplest of the formulas: “i^{θ}”.

We want to find the value “c” in the following:
$c^{i\theta} = i^{\theta}$

In other words, we want to find the base that, when raised to an Imaginary power, treats “θ” as an angle in quarter-circle angle units. To do this, we will make both halves refer to an actual point around a unit-radius circle. We will use the point at “i^1”, which is a quarter of the way around the circle. This is an arbitrarily chosen point, but it makes the maths simpler.

Therefore, we have:
$c^{1i} = i^1$
... where “c” is the unknown base. This would normally be written as:
$c^i = i$

We can take the i^{th} root of each side of the equation, and rephrase the formula as:
$c = \sqrt[i]{i}$

In English, this says that “c” is equal to the i^{th} root of i. We could resort to using a calculator that can work with Complex numbers, but for now, we will say that we do not know how to solve $\sqrt[i]{i}$.

We know that $\sqrt[i]{i}$ raised to the power of “iθ” will indicate the position of a point on a unit-radius circle at an angle of “θ” quarter-circle angle units. We can also say that:
$(\sqrt[i]{i})^{i\theta} = \cos\theta + i \sin\theta$
... where “θ” is an angle in quarter-circle angle units, and Cosine and Sine are working in quarter-circle angle units.

Note that the above equivalence is nothing new. We know that “$i^\theta = \cos\theta + i \sin\theta$” when “θ” is an angle based on a system that divides a circle up into 4 parts, and when Sine and Cosine are working in that system. [If “i^θ” marks the position of a point on a unit-radius circle at an angle of “θ” quarter-circle angle units, then it must be the case that that point can also be described using “$\cos\theta + i \sin\theta$” when Cosine and Sine are working in the same system.]

From the equivalence “$i^\theta = \cos\theta + i \sin\theta$”, it should be clear that *any* formula that means the same thing as “i^θ”, no matter how it is phrased, will *always* be equivalent to “$\cos\theta + i \sin\theta$” in a 4 division angle system. Similarly, if we alter the “i^θ” formula so that “θ” can be used to represent angles in other angle systems, then that adjusted formula will *always* be equivalent to “$\cos\theta + i \sin\theta$” in that other angle system.

With the knowledge we have at the moment, we cannot solve most cases of $(\sqrt[i]{i})^{i\theta}$, but as this explanation progresses, we will find a way.

To illustrate the base being used to represent points on a unit-radius circle in practice, here are some simple examples that can be solved without a calculator. These are not the most thoroughly representative examples, but they illustrate the concept simply.

Example 1

The point at 2 quarter-circle angle units on a unit-radius circle can be represented with: $(\sqrt[i]{i})^{2i}$

The i^{th} root of something, when raised to the power of “i” will cause the root and the power to cancel out. Therefore, we are left with:
$(i)^{2}$
... which is:
−1.

The point’s position is at “−1 + 0i”.

Example 2

The point at 4 quarter-circle angle units can be represented with: $(\sqrt[i]{i})^{4i}$.

Again, the i^{th} root and the power of “i” cancel out. We are left with:
$(i)^{4}$
$= (-1)^{2}$
= 1.

The point’s position is at “1 + 0i”.

Example 3

The point at 0.5 angle units can be represented with: $(\sqrt[i]{i})^{0.5i}$.

The i^{th} root and the power of “i” cancel out again. We are left with this:
$(i)^{0.5}$

We know where this point is on a circle – it is at an angle of 0.5 quarter-circle angle units, which is the same as 45 degrees. Therefore, the point’s position is at:
“0.7071 + 0.7071i”.

Although we can use “($\sqrt[i]{i}$)$^{i\theta}$” to identify points, we cannot yet make it into a Real number raised to an Imaginary power, as we do not yet know enough to solve the ith root of “i”. We will come back to this later.

Whole-circle angle units

For whole circle-angle units, the formula using powers of “i” was“$i^{4\theta}$”. To turn this into a formula that has a Real base and an Imaginary exponent, we need to find out what “c” is in this equation:

$c^{i\theta} = i^{4\theta}$

To solve this, we will use the value of “θ” that represents the equivalent of 180 degrees – in other words, 0.5 whole-circle angle units. Again, the choice of this number is completely arbitrary. We end up with this:

$c^{0.5i} = i^{4*0.5}$

$c^{0.5i} = i^{2}$

$c^{0.5i} = -1$

c = $\sqrt[0.5i]{-1}$

If we treat the radicand, −1, as being $(-1)^1$, we can double the index of the root, 0.5i, and halve the exponent of the radicand, 1, at the same time. The overall meaning will be the same:

$\sqrt[0.5i]{-1}$

... is the same as:

$\sqrt[0.5i]{(-1)^1}$

... which is the same as:

$\sqrt[0.25i]{(-1)^{0.5}}$

... and because, $(-1)^{0.5}$ is the square root of −1, which is “i”, this is:

$\sqrt[0.25i]{i}$

Therefore, we can say:

c = $\sqrt[0.25i]{i}$

We can identify the position of any point around a unit-radius circle by giving $\sqrt[0.25i]{i}$ raised to the power of “iθ”, where “θ” is an angle in whole-circle angle units:

($\sqrt[0.25i]{i}$)$^{i\theta}$

We can also say that:
$(\sqrt[0.25i]{i})^{i\theta}$ = cos θ + i sin θ"
... where Cosine and Sine are working in whole-circle angle units, and "θ" is being treated as an angle in whole-circle angle units.

Example 1

As an example of this working, the point at 0.5 whole-circle angle units can be represented with:
$(\sqrt[0.25i]{i})^{i0.5}$

This ends up as −1, which means that the point is at "−1 + 0i".

Example 2

The point at 0.25 whole-circle angle units can be represented with:
$(\sqrt[0.25i]{i})^{i0.25}$

This ends up as $\sqrt{-1}$, or "i", so the point is at "0 + 1i".

Example 3

The point at 0.75 angle units can be represented with:
$(\sqrt[0.25i]{i})^{i0.75}$

This can be solved with the following steps:
$(\sqrt[0.25i]{i})^{i0.75}$
= $(\sqrt[1]{i})^3$
= $(i)^3$
= i * i * i
= −1 * i
= −i

Therefore, the point is at "0 – 1i".

Degrees

To rebase the formula for degrees, which is “$i^{\theta/90}$”, we need to find “c” in this equation:
$c^{i\theta} = i^{\theta/90}$

To solve this, we will use the value of “θ” that represents the equivalent of 180 degrees, which, obviously, is 180 degrees. Therefore, we end up with this:
$c^{180i} = i^{180/90}$

This reduces to:
$c^{180i} = i^2$
$c^{180i} = -1$
$c = \sqrt[180i]{-1}$
$c = \sqrt[90i]{i}$

We can use this as a base to the power of “iθ” where “θ” is an angle in degrees. The formula of $(\sqrt[90i]{i})^{i\theta}$ gives the position of a point on a unit-radius circle at an angle of “θ” degrees. That point’s position can also be given by “cos θ + i sin θ” when “θ” is in degrees, and Cosine and Sine are working in degrees. To put this another way:
$(\sqrt[90i]{i})^{i\theta} = \cos\theta + i \sin\theta$
... when Cosine and Sine are working in degrees.

Example 1

As an example of the formula in use, the point at 270 degrees can be represented with:
$(\sqrt[90i]{i})^{i270}$

We can reduce the root and power by thinking of them as multiples of 90i, and we end up with:
$(\sqrt[1]{i})^3$
$= (i)^3$
$= -i$

Therefore, the point is at “0 − 1i”.

Example 2

The point at 22.5 degrees can be represented with:
$(\sqrt[90i]{i})^{i22.5}$

This ends up as:
$(\sqrt[4]{i})^{1}$
$= \sqrt[4]{i}$
$= i^{0.25}$

We know that "$i^{0.25}$" is the point at 0.25 quarter-circle angle units. This is the point at 0.25 ÷ 4 = 0.0625 of the way around the circle, which is also the point at 22.5 degrees [which we also knew because that is the angle that we started with in this example]. Therefore, this point is at "cos 22.5 + i sin 22.5", where Cosine and Sine are working in degrees. This is "0.9239 + 0.3827i".

Radians

To rebase the formula for radians, "$i^{2\theta/\pi}$", we need to find "c" in this equation:
$c^{i\theta} = i^{2\theta/\pi}$

To do this, we will use the value of "θ" that represents the equivalent of 180 degrees, which is the angle of π radians. We end up with this:
$c^{i\pi} = i^{(2*\pi)/\pi}$

This can be simplified as:
$c^{i\pi} = i^{(2\pi)/\pi}$
$c^{i\pi} = i^{2}$
$c^{i\pi} = -1$
$c = \sqrt[i\pi]{-1}$
$c = \sqrt[0.5i\pi]{i}$

We can use this as a base to the power of "iθ" where "θ" is an angle in radians. The formula $(\sqrt[0.5i\pi]{i})^{i\theta}$ gives the position of a point on a unit-radius circle at an angle of "θ" radians. That point can also be identified using "cos θ + i sin θ" when "θ" is an angle in radians, and Cosine and Sine are working in radians:
$(\sqrt[0.5i\pi]{i})^{i\theta} = \cos\theta + i \sin\theta$
... when Cosine and Sine are working in radians.

It is worth clarifying here that all these equivalences with Cosine and Sine are *only* true for the particular angle system for the chosen base. Therefore, it is not the case that the above equation is true for any angle system apart from radians. If Cosine and Sine were working in, say, degrees, then the above equation would not work.

Example 1

The point on a unit-radius circle at an angle of π radians is situated at:
$(\sqrt[0.5i\pi]{i})^{i\pi}$
... which is:
$(\sqrt[0.5]{i})$
... which is:
i^2
... which is:
−1, which is the point at “−1 + 0i”.

Example 2

The point at an angle of 0.5π radians is situated at:
$(\sqrt[0.5i\pi]{i})^{0.5\pi i}$
... which is:
i, which is the point at “0 + 1i”.

Example 3

The point at an angle of 1.75π radians is situated at:
$(\sqrt[0.5i\pi]{i})^{1.75\pi i}$
... which is:
$(\sqrt[1]{i})^{3.5}$
... which is:
$i^{3.5}$

From what we know about “i” raised to powers, this will be the point on a unit-radius circle at an angle of 3.5 quarter-circle angle units. This point is 3.5 ÷ 4 = 0.875 of the way around a circle, which is the point at 0.875 * 360 = 315 degrees. By thinking about the circle, we know that this is “0.7071 – 0.7071i”.

Chapter 4: Converting the bases

Our original exponentials with “i” as a *base* were:

- For a unit-radius circle divided into 4 parts: “i^{θ}”
- For a unit-radius circle as one part: “$i^{4\theta}$”
- For a unit-radius circle in degrees: “$i^{\theta/90}$”
- For a unit-radius circle in radians: “$i^{2\theta/\pi}$”

Our new exponentials with “i” as an *exponent* are:

- For a unit-radius circle divided into 4 parts: “$(\sqrt[i]{i})^{i\theta}$”
- For a unit-radius circle as one part: “$(\sqrt[0.25i]{i})^{i\theta}$”
- For a unit-radius circle in degrees: “$(\sqrt[90i]{i})^{i\theta}$”
- For a unit-radius circle in radians: “$(\sqrt[0.5i\pi]{i})^{i\theta}$”

With no actual evidence, we will presume that these new bases can all be converted into Real numbers, which is something that would make them easier to use. We could just use a calculator that can work with Complex numbers to solve these bases, but we need to find a way that will show the connection in a more obvious way, so that we can see what is happening.

One fact about all of these exponentials is that they can all be used to refer to the same points around the circle. Therefore, they can all be equivalent if values that refer to the same point are entered into them.

For example, if we put the equivalent of an angle of 180 degrees into each exponential formula, they will all result in −1 + 0i, which is −1:

$(\sqrt[i]{i})^{i2} = -1$

$(\sqrt[0.25i]{i})^{0.5i} = -1$

$(\sqrt[90i]{i})^{i180} = -1$

$(\sqrt[0.5i\pi]{i})^{i\pi} = -1$

... which means that all of *these* formulas are the same as each other. However, this does not help solve anything because all of the formulas were derived from the same original formula. Using these will, at best, mean we end up with a conclusion such as “1 = 1”.

If we put the equivalent of an angle of 360 degrees into each formula, all the formulas will result in 1 + 0i, which is 1:

$(\sqrt[i]{i})^{i4} = 1 + 0i$

$(\sqrt[0.25i]{i})^{i} = 1 + 0i$

$(\sqrt[90i]{i})^{i360} = 1 + 0i$

$(\sqrt[0.5i\pi]{i})^{i2\pi} = 1 + 0i$

... which means that all of these are the same as each other too. What is significant about this idea is that in each case, we have an exponential that is equal to the full circle. For this to be so, and presuming Imaginary exponentials follow the rules of non-Imaginary exponentials, then it must be the case that the smaller the base, the larger the exponent must be, and the larger the base, the smaller the exponent must be. We can put the exponentials in order of the size of their exponent, from smallest to largest, like so:

$(\sqrt[0.25i]{i})^{i} = 1 + 0i$

$(\sqrt[i]{i})^{i4} = 1 + 0i$

$(\sqrt[0.5i\pi]{i})^{i2\pi} = 1 + 0i$

$(\sqrt[90i]{i})^{i360} = 1 + 0i$

For all of these exponentials to result in "1 + 0i", it must be the case that $(\sqrt[0.25i]{i})$ is a larger number than $(\sqrt[i]{i})$, which in turn is a larger number than $(\sqrt[0.5i\pi]{i})$, which is a larger number than $(\sqrt[90i]{i})$. We can also tell this is so by how the index of each root in the list is higher than the one before it, so the roots as a whole must be lower. There must be an inverse relationship between the bases and the exponents for them all to be equal to the same amount.

Given *that*, and given the fact that we are presuming the bases can all be expressed with Real numbers, if we remove the "i" from the exponents, the formulas would all be equal to the same Real number. We will call this number "S" for "the Special number". In other words:

$(\sqrt[0.25i]{i})^{1} = S$

$(\sqrt[i]{i})^{4} = S$

$(\sqrt[0.5i\pi]{i})^{2\pi} = S$

$(\sqrt[90i]{i})^{360} = S$

... where "S" is the same Real number in each case.

This means that the number "S" in the Real world is an equivalent to the full circle in the Complex plane. It is the value that links an exponential with a Real base and a Real exponent to a circle represented by a Real base and an Imaginary exponent.

Given that "S" represents the non-Complex equivalent to a full circle, and that "S" is calculated with exponential numbers, it will be the case that $\sqrt{S}$ will be the non-Complex equivalent to half a circle (180 degrees), $\sqrt[4]{S}$ will be the equivalent to quarter of a circle (90 degrees), $\sqrt[8]{S}$ will be the equivalent to an eighth of a circle (45 degrees), and so on.

This means that any exponential with a Real base and a Real exponent, that is equal to, say, $\sqrt{S}$, will, if the exponent is multiplied by "i", be equal to a point on the circumference of a unit-radius circle on the Complex plane at 180 degrees.

In other words:

- If $c^x = S$, then c^{ix} identifies the position of a point on a unit-radius circle at an angle of 360 degrees.
- If $c^x = S$, then $c^{0.5ix}$ identifies the position of a point on a unit-radius circle at an angle of 180 degrees.
- If $c^x = S$, then $c^{0.75ix}$ identifies the position of a point on a unit-radius circle at an angle of 270 degrees.

Finding S

If we can find "S", we will be able to calculate the bases for all our new exponentials, and for any other exponentials we care to have, and with a minimum of effort.

We know that:

$c^x = S$

...where "c" is a base, and "x" is the number of divisions in a circle. Therefore, if we knew what "S" is, we would be able to find the base for any system of dividing up a circle by using this equation:

$c = \sqrt[x]{S}$

... where "x" represents the desired divisions in a circle, and "c" is the base that will produce an exponential, with an Imaginary exponent, that will operate according to that angle system.

We know that:
$(\sqrt[0.25i]{i})^1 = S$
$(\sqrt[i]{i})^4 = S$
$(\sqrt[0.5i\pi]{i})^{2\pi} = S$
$(\sqrt[90i]{i})^{360} = S$
[Note how these all have non-Imaginary exponents]

Therefore, we know that “S” is exactly $\sqrt[0.25i]{i}$. However, this is of no use to us as we cannot find the $0.25i^{th}$ root of “i” using the information that we have so far.

We also know that “S” is $(\sqrt[0.5i\pi]{i})^{2\pi}$. This comes from the formula for a circle divided into 2π portions: $(\sqrt[0.5i\pi]{i})^{2\pi i}$. It might be obvious from the subject of this entire document that the base for this will be the number “e”. Unfortunately, I do not know a way of working this out without already knowing that “e” works in an identical way to $\sqrt[0.5i\pi]{i}$. If we already have experience of the behaviour of “e”, we can tell that “e” is the base, and it is possible to test that it is correct. We can be sure that the base *is* “e” by examining the behaviour of “e” raised to Imaginary powers (which we will do in the next chapter), and seeing how it behaves in an identical way to $\sqrt[0.5i\pi]{i}$ raised to Imaginary powers. If we do not guess that “e” is the base, there does not seem to be any way (that I can think of) of calculating what the base must be. Anyway, if we use “e” as the base, everything else will fall into place.

In the second from last paragraph, I said that we could not, yet, solve a calculation such as $\sqrt[0.25i]{i}$. The reason for this is that all the methods for solving such a calculation involve prior knowledge of “$e^{i\theta}$”. Therefore, it would have been cheating to use those methods when we had not shown that “e” was the base. A calculator that can work with Complex numbers could have found the result, but it would have (presumably) used prior knowledge of “$e^{i\theta}$”.

As we know that “e” is the base, we can say that:
$e = \sqrt[0.5i\pi]{i}$
... or, putting this another way:
$\sqrt[0.5i\pi]{i} = 2.718281828459...$

Chapter 5: Proof that e is the base for radians

In this chapter, I will show that "e" is the base for the exponential, "$c^{i\theta}$", where "θ" is an angle in radians. As I said in the previous chapter, I do not know a method of discovering that "e" is the base without already knowing that the behaviour of "e" matches the behaviour of $\sqrt[0.5i\pi]{i}$. However, once we guess that "e" is the base, we can see how "$e^{i\theta}$" has all the characteristics that we expect.

The first step in the proof is to calculate "e^{i}". Fortunately, there is a simple way to do this that is based around the concept of bank account interest.

Compound bank interest

A bank account that pays compound interest is one that pays interest based on the amount of money in the account, and then, if we keep that interest in the account, pays the next amount of interest based on the new total in the account.

Supposing we never spend any money, if we start with £100 in a bank account and we are paid 1% interest a year, at the end of the first year, we will get a payment of £1. Our new balance will be £101. At the end of the second year, we will earn 1% interest on that £101. This works out as £1.01. We now have £102.01 in the account. At the end of the third year, we will earn 1% interest on that £102.01 and earn £1.0201, which, if our bank works in hundredths of pennies, will mean that we have £103.0301 in our account.

Supposing we did not have compound interest, the interest each year would only be on the original sum of money. In that case, at the end of the first year, we would earn 1% of £100. At the end of the second year, we would still earn 1% of £100. At the end of the third year, we would still earn 1% of £100. After 3 years, we would only have earned £3 in total compared to the £3.0301 we earned when we had compound interest.

Compound interest means that we earn more money overall.

Monthly compound interest

In the above example, regardless of whether we have compound interest or not, after the first year, we will get £1 interest.

If the bank decided to pay us the interest once a *month* instead of once a year, they would split the annual interest payment into 12 parts, and pay one part every month. In this way, if the interest rate were 1%, we would earn 1% ÷ 12 ≈ 0.08333% interest every month. This is the same as receiving 0.0008333 times the total amount in the account each month. If we have £100 invested, then at the end of the first month, we would receive 0.0008333 * £100 = £0.08333.

If we did not have compound interest, monthly interest payments would not be any different to having the interest paid annually in one go – we would earn £0.08333 each month and end up with £1 interest at the end of the year. However, if we *did* have compound interest, we would receive interest calculated on the interest we had earned each month, and end up with more money.

We would earn 0.0008333 times the amount in the bank account each month. At the end of each month, the total would be 1.000833 times as much as it was the month before.

At the end of the first month, we would receive 0.0008333 * £100 = £0.08333 interest. The total in the bank account would be 1.0008333 * £100 = £100.08333.

At the end of the second month, we would receive 0.0008333 * £100.08333 = £0.083402 interest. At this point, in the bank account, we would have a total of 1.0008333 * £100.08333 = £100.1667.

At the end of the third month, the total would be £100.2502
At the end of the fourth month: £100.3338
At the end of the fifth month: £100.4174
At the end of the sixth month: £100.5010
At the end of the seventh month: £100.5848
At the end of the eighth month: £100.6686
At the end of the ninth month: £100.7525
At the end of the tenth month: £100.8365
At the end of the eleventh month: £100.9205
At the end of the twelfth month: £101.0046

By having the interest paid monthly, over the course of a year, we would end up receiving £0.0046 more than if the interest were paid just once a year.

Formulas for interest

It is easiest to think of the fraction that a percentage represents instead of thinking of the actual percentage rate. Therefore, in this section, I will give the yearly percentage rates in terms of their fractions. This will be the percentage rate divided by 100. In other words, I will say 0.05 instead of 5%, I will say 0.12 instead of 12%, and so on. I will call this fraction the "annual interest fraction".

We can make a formula to work out the total amount of money in the bank account at the end of twelve months. The total is equal to:

original amount * (1 + (annual interest fraction ÷ 12))^{12}

We can express this more mathematically as:
$y = a * (1 + (f \div 12))^{12}$
... where:

- "y" is the resulting total after one year's interest.
- "a" is the original amount.
- "f" is the annual interest fraction.
- "12" is the number of months in one year.

The annual interest fraction is the annual interest rate divided by 100. In other words, it is the amount that we would multiply by the original total to work out that year's interest if it were going to be paid in one go.

That number is then divided by 12, which is the number of months in one year, and also the number of payments in one year. This results in the fraction for one monthly payment.

We add 1 to this so we include the original total in the result. If we did not do this, we would just end up with the amount paid instead of the original amount added to the amount paid.

We then multiply this by itself for every month we want to count. If we wanted two months, we would square it. If we wanted three months, we would cube it. As we want the result for a full 12 months of interest payments, we raise it the power of 12.

Then, we multiply that by the original amount in the bank account.

The result is the original total added to the paid interest over the full year.

A more general formula

We can make the formula relate less to years and months, so it becomes more general:

$y = a * (1 + (f \div n))^n$
... where:

- “y” is the total after one overall period of time.
- “a” is the original amount in the bank account.
- “f” is the interest fraction for one overall period.
- “n” is the number of payments made in one overall period.

Note that this formula is now independent to time. If the number of payments for an overall period is 12, then it does not matter if the overall period is years, months, seconds or anything. The result will still be the same.

For our original example of 1% interest each year on a £100 initial investment, with the interest paid monthly, we can fill in the formula like so:
Total after one year = $100 * (1 + (0.01 \div 12))^{12}$
≈ £101.004596
... and this matches what we had before.

More intervals

We will look at what would happen if we convinced the bank to pay interest every day instead of every month. The total in the bank account after one year would be:
$100 * (1 + (0.01 \div 365))^{365}$ ≈ £101.005003.

We have ended up receiving about 0.5 pence more than if the interest were only paid once a year. However, this is only an extra 0.0004 pence more than if the interest were paid every month.

If the bank paid compound interest every hour, the formula would be:
$100 * (1 + (0.01 \div 8760)^{8760}$ ≈ £101.005016.

The result is only fractionally more than before.

If the bank paid interest every second, the formula would be:
$100 * (1 + (0.01 \div 31{,}536{,}000)^{31{,}536{,}000}$ ≈ £101.005017.

The extra amount is negligible.

It is clear that after a certain point, increasing the payment intervals does not produce a significantly larger result. We get diminishing returns on the number of interest payments. There is no way to trick the bank into paying a lot more money.

Finding the limit on interest

If we ignore the original bank balance of £100, and just concentrate on the formula, we can examine the limit more closely. The part of the formula that is multiplied by the original balance is this:
$(1 + (0.01 \div n))^{n}$
... where "n" is the number of interest payments.

We can put in different values of "n" to see what happens:
If "n" is 1, the result is 1.01
If "n" is 10, the result is 1.01004512021025
If "n" is 100, the result is 1.01004966209288
If "n" is 1000, the result is 1.01005011658
If "n" is 1,000,000, the result is 1.0100501669728

It turns out that we are just getting closer and closer to a number that begins 1.010050...

We will try different annual interest rates over a very large number of interest payments. We will say there are a million interest payments, so we get a good approximation of the limits for each interest rate. The general formula for this will be:
$(1 + (f \div 1{,}000{,}000)^{1{,}000{,}000}$
... where "f" is the interest rate fraction.

We will start with an interest rate of 0.01%, which is an interest fraction of 0.0001. We will then move up to higher and higher interest rates.

Annual Interest at 0.01%: $(1 + (0.0001 \div 1{,}000{,}000))^{1{,}000{,}000} \approx 1.00010001$
Annual Interest at 0.1%: $(1 + (0.001 \div 1{,}000{,}000))^{1{,}000{,}000} \approx 1.00100050$
Annual Interest at 1%: $(1 + (0.01 \div 1{,}000{,}000))^{1{,}000{,}000} \approx 1.01005017$
Annual Interest at 10%: $(1 + (0.1 \div 1{,}000{,}000))^{1{,}000{,}000} \approx 1.05170913$
Annual Interest at 100%: $(1 + (1 \div 1{,}000{,}000))^{1{,}000{,}000} \approx 2.71828047$
Annual Interest at 1000%: $(1 + 10 \div 1{,}000{,}000)^{1{,}000{,}000} \approx 22{,}025.36450783$

The above results are all the approximate values that the interest payments will converge to for various annual interest rates. If we increased the number of interest payments, the accuracy for each result would be higher.

There is clearly a pattern in these numbers, but it is hard to tell what it is at first glance. You might be able to figure it out by playing around with a calculator. It turns out that every result is actually the number 2.718281828... raised to the power of the interest fraction.

In other words:

- When the interest rate is 0.01%, the fraction is 0.0001 and the result in the above table is $2.71828^{0.0001} \approx 1.00010001$
- When the interest rate is 0.1%, the fraction is 0.001 and the result is $2.71828^{0.001} \approx 1.00100050$
- When the interest rate is 1%, the fraction is 0.01 and the result is $2.71828^{0.01} \approx 1.01005017$
- When the interest rate is 10%, the fraction is 0.1 and the result is $2.71828^{0.1} \approx 1.10517092$
- When the interest rate is 100%, the fraction is 1 and the result is $2.71828^{1} \approx 2.71828183$
- When the interest rate is 1000%, the fraction is 10 and the result is $2.71828^{10} \approx 22{,}026.4658$

The number 2.718281828... is of course the number generally known as “e”.

For any interest fraction, if a single interest payment period is split up into an infinite number of smaller periods and paid with compound interest, the maximum interest achievable will be 2.71828... raised to the power of that fraction.

In other words, if we have £100 in a bank account and an annual interest rate of 1%, it does not matter whether the bank pays the annual interest split up into months, days, minutes, seconds, or microseconds, the most money we will ever have at the end of the year will be:
$100 * e^{0.01} = £101.00501671$

Similarly, if we have £1 in a bank account, and an annual interest rate of 100%, even if the interest were paid at an infinite number of intervals throughout the year, the most money we can ever have at the end of the year will be:
$1 * e^{1} = £2.718281828$

Formulas to approximate e

Given all the above, we have a formula to approximate "e":
$e \approx (1 + (1 \div n))^n$
... where "n" is any very large number. The bigger it is, the more accurate the result will be.

We also have a formula to approximate "e" raised to any power:
$e^x \approx (1 + (x \div n))^n$
... where "x" is the number to which we want to raise "e", and "n" is any very large number.

e to the power of ix

Given that we now have a formula to approximate "e" raised to any power, we can use it to find out what happens if we raise "e" to the power of "i".

We will use the $(1 + (x \div n))^n$ formula to calculate "e^i", which, to make things clearer, we will write as "e^{1i}".

We put "1i" into the formula and choose a large number for "n":
$(1 + (1i \div 10{,}000))^{10{,}000}$
$= (1 + (0.0001i))^{10{,}000}$
$= (1 + 0.0001i)^{10{,}000}$

This is a Complex number raised to the power of 10,000: $(1 + 0.0001i)^{10{,}000}$

We can solve this by multiplying "1 + 0.0001i" by itself 10,000 times.

First, we multiply "1 + 0.0001i" by itself:
$(1 + 0.0001i) * (1 + 0.0001i)$
$= 0.0001i + 0.0001i + 1 + (0.0001i)^2$
$= 0.0002i + 1 - 0.00000001$
$= 0.0002i + 0.99999999$
$= 0.99999999 + 0.0002i$

Then, we multiply that result by "1 + 0.0001i":
$(0.99999999 + 0.0002i) * (1 + 0.0001i) = 0.99999997 + 0.000299999999i$

We then multiply that result by “1 + 0.0001i”, and continue like this until we have achieved “1 + 0.0001i” multiplied by itself 10,000 times. It would be possible to do this on a piece of paper, but it would take a very long time. It is quicker to write a small computer program to do it [or to use a calculator that can work with Complex numbers].

Eventually, we will end up with “0.5403 + 0.8415i” (to 4 decimal places).

What is interesting about this Complex number is that it indicates the position of a point exactly 1 unit from the origin of the axes at an angle of exactly 1 radian (57.2958 degrees). If a point is at an angle of 1 radian and is exactly 1 unit from the origin of the axes, then it is also the point at “cos 1 + i sin 1” when Cosine and Sine are working in radians.

We could calculate countless other Imaginary exponents of “e” in the same way, and in each case, we would find that the result was a point 1 unit away from the origin of the axes at an angle equal to the non-Imaginary part of the exponent. In other words, if we have “$e^{i\theta}$”, then it indicates the position of a point 1 unit from the origin of the axes at an angle of “θ” radians. We can also say that “$e^{i\theta}$” indicates the position of a point *on the edge of a unit-radius circle* at an angle of “θ” radians.

We can make the general statement that:
“$e^{i\theta} = \cos\theta + i \sin\theta$”
... when “θ” is an angle in radians and Cosine and Sine are working in radians. [To be pedantic, it does not matter whether “θ” is intended to be an angle in radians or not because it will be treated as an angle in radians anyway by the fact it is an Imaginary exponent of “e”.]

From all of this, we can see that “e” raised to an Imaginary power works in a similar way to “i” raised to a power. The values “$e^{i\theta}$” and “i^{θ}” both identify a point on a unit-radius circle based on treating “θ” as an angle. Imaginary powers of “e” work in radians; powers of “i” work in quarter-circle angle units.

In addition to this, in a similar way to how a multiplication by “i^{θ}” rotates a point anticlockwise by “θ” quarter-circle angle units, so does a multiplication by “$e^{i\theta}$” rotate a point anticlockwise by “θ” radians.

What is most important here is that now we know for sure that $(\sqrt[0.5i\pi]{i})^{i\theta}$ must be equal to “$e^{i\theta}$” because it works in exactly the same way. Therefore, $\sqrt[0.5i\pi]{i}$ must be equal to “e”.

Note that I could have explained all of the above using the various Taylor series for “e”, Sine, and Cosine, and we would have come to the same conclusion. However, I have noticed that some people treat the way that the Taylor series for “e” reveals the connection between “$e^{i\theta}$” and “cos θ + i sin θ” as if it were something to do with the Taylor series itself, rather than a fact of reality. The Taylor series is just *revealing* the connection – the connection is not a result of the Taylor series. Also, the reasons for why the Taylor series for “e” is how it is are much harder to understand than compound interest.

Chapter 6: Calculating and using "S"

We have seen that "$e^{i\theta}$" can identify points around a unit-radius circle that are at an angle of "θ" radians, and we can now say that $\sqrt[0.5i\pi]{i}$ is equal to "e". Once we know this, everything else can be easily solved. We can now calculate "S". The number "S" will be the result of "$e^{2\pi}$", which is 535.491655524765 [to 12 decimal places].

Note that there is nothing necessarily special about "$e^{2\pi}$" in this situation. The breakthrough in finding the value of "S" was through finding another way to calculate a base. In this case, we knew a way of calculating "$e^{2\pi i}$", and so that let us calculate "S". If there were some other formula for calculating the base for circles divided up into other numbers of pieces (e.g. 360 pieces, 4 pieces, 112 pieces and so on), then that would have worked just as well. The number 535.4917 could be said to be equal to countless other exponentials, and it is a matter of choice that we say it is "$e^{2\pi}$" as opposed to anything else. [Although, on the other hand, "e" and "π" are probably the only number pair for which we already have names and symbols, and they have an advantage in that their values are usually programmed into calculators.]

It would be interesting to know if there are other ways of calculating "S" that do not require "e" and "π".

Now that we know S

From knowing "S", we can calculate the bases for our other angle types using the equation:
$c = \sqrt[x]{S}$
... where:

- "c" is the base that we want to find out.
- "x" is the number of divisions into which the circle has been divided for the particular angle system that we are dealing with.

Quarter-circle angle units

A circle that has been divided into 4 portions will have the base:
$\sqrt[4]{535.4917}$ = 4.810477380965 [to 12 decimal places].

Therefore, $4.8105^{i\theta}$ gives the position of a point on a unit-radius circle at an angle of "θ" quarter-circle angle units.

We can also say that "$4.8105^{i\theta} = \cos\theta + i\sin\theta$", when Cosine and Sine are working in the quarter-circle angle unit system.

Given that our provisional base for quarter-circle angle units was $\sqrt[i]{i}$, we now know that:
$\sqrt[i]{i}$ = 4.81047738

By finding "S", we have become able to solve what looked like a very complicated calculation. We can say that:
$\sqrt[i]{i} = 4.8105 = \sqrt[4]{535.4917} = \sqrt[4]{e^{2\pi}}$

Looking back at our first system of indicating points around a unit-radius circle using "i^{θ}", we can say that:
$i^{\theta} = 4.8105^{i\theta}$

Whole-circle angle units

A circle that has been "divided" into 1 portion will have the base:
$\sqrt[1]{535.4917}$
= 535.4917

Therefore, $535.4917^{i\theta}$ gives the position of a point on a unit-radius circle at an angle of "θ" whole-circle angle units. [We could also phrase this by saying that if "θ" is a fraction of a circle, then $535.4917^{i\theta}$ gives the position of a point on a unit-radius circle at that fraction around the edge.]

We can also say that "$535.4917^{i\theta} = \cos\theta + i\sin\theta$", when Cosine and Sine are working in whole-circle angle units.

We can also say that:
$\sqrt[0.25i]{i} = 535.4917 = e^{2\pi} = 4.8105^{4}$

Degrees

A circle that has been divided into 360 portions will have the base:
$\sqrt[360]{535.4917}$
= 1.017606491206 [to 12 decimal places].

Therefore, $1.0176^{i\theta}$ gives the position of a point on a unit-radius circle at an angle of “θ” degrees.

We can also say that “$1.0176^{i\theta} = \cos\theta + i\sin\theta$”, when Cosine and Sine are working in degrees.

We can also say that:
$\sqrt[90i]{i} = 1.0176$
... and:
$1.0176^{360} = 535.4917^{1} = e^{2\pi} = 4.8105^{4}$

Radians

As we already know, a circle that has been divided into 2π portions will have as its base:
$\sqrt[2\pi]{535.4917}$
= 2.718281828459 [to 12 decimal places].
... which is the number “e”.

Therefore, “$e^{i\theta}$” gives the position of a point on a unit-radius circle at an angle of “θ” radians. It is also the case that “$e^{i\theta} = \cos\theta + i\sin\theta$”, when Cosine and Sine are working in radians.

We can also say that ($\sqrt[0.5i\pi]{i}$) = e.

Table of bases and exponents

There are countless ways to divide up a circle, and therefore, there are countless exponentials that can indicate the position of points around a unit-radius circle. The rules for calculating the bases and angle systems are as so:

We can calculate the base for an angle system using the formula:
c = $\sqrt[x]{535.4917}$
... where:

- “c” is the base that we want to find.
- “x” is the number of portions into which the circle has been divided, which we know.
- 535.4917 is “$e^{2\pi}$”.

We can calculate the angle system for a particular base by using the formula:
“x = $\log_c$ (535.4917)”
... where:

- “x” is the number of portions into which the circle has been divided, which is the value we want to find.
- “c” is the base, which we know.
- 535.4917 is “$e^{2\pi}$”.

On the next page is a table of a few examples of bases and angle systems. A value taken from first column, when raised to the power of the corresponding value from the second column, will result in 535.4917. The same exponential with the exponent multiplied by “i” will indicate a point on a unit-radius circle at an angle that is equivalent to 360 degrees.

Base for a particular circle division	**Number of divisions in the circle**	
535.491655524765	1	
23.140692632779	2	This base is “Gelfond's constant”.
10.089090581019	2.7182...	This circle is divided into “e” divisions.
10	2.728752707684	
8.120527396670	3	
7.389056098931	3.1415... (π)	This base is e^2 and the circle is divided into π pieces. This base is more obvious if you consider that $(e^2)^\pi = e^{2\pi}$.
4.810477380965	4	This is the base for quarter-circle angle units.
4.304530324517	4.304530324517	The number of divisions in this circle is the same as the base. In other words, $4.3045^{4.3045i}$ indicates a point on a unit-radius circle’s edge at the equivalent of 360 degrees, in a system that divides a circle into 4.3045 pieces.
3.513585624286	5	
3.1415... (π)	5.488792932594	
2.718281828459 (e)	2π (6.28...)	This is $e^{2\pi}$.
2	9.064720283654	
1.874456087585	10	
1.369107777062	20	
1.064847773329	100	
1.017606491206	360	This is the base for working in degrees.
1.006302965923	1000	
1	∞	This circle would need to be divided into an infinite number of pieces.

There might be well known number pairs within the possible bases and circle divisions. If so, this would mean that they also have some connection to “e” and “π”.

4.304530324517

Of the numbers in the table, one that stands out is 4.30453032 (given to 8 decimal places). With this number:
$4.30453032^{4.30453032} = 535.49165552$
... and:
$4.30453032^{4.30453032i}$ indicates the position of a point at the equivalent of 360 degrees on a unit-radius circle.

We can also say that:
$4.30453032^{4.30453032i} = 1 + 0i$
... which can also be written as:
$4.30453032^{4.30453032i} = 1$.

This is an interesting equation because it means that 4.30453032 is the solution to:
$x^{xi} = 1$
... and:
$x^x = e^{2\pi}$

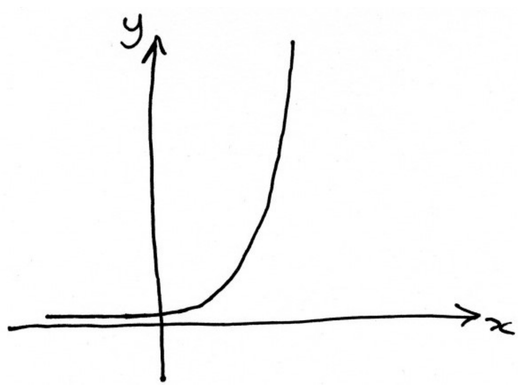
Negative divisions

The formula “$c = \sqrt[x]{S}$” still works for negative values of “x”. Therefore, it is possible to divide a circle into a negative number of portions, and still have a valid exponential. For example, if we want the circle divided into −2 divisions, the base we need is 0.043213918264 [to 12 decimal places]. From this we can say that $0.043213918264^{-2} = S$. Furthermore, 0.043213918264^{-2i} represents a point on a unit-radius-circle that is at an angle equivalent to 360 degrees.

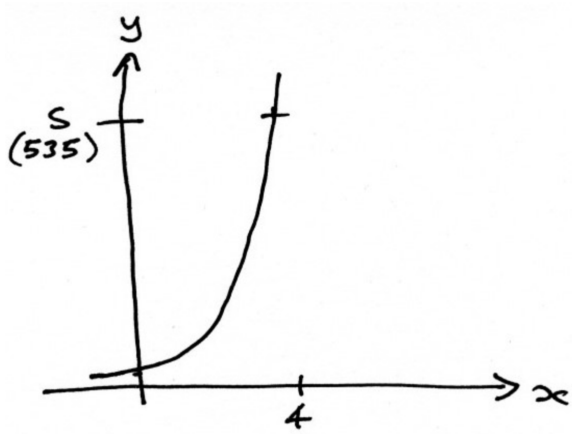
The base for a negative number of circle divisions is the reciprocal of the base for that same number of circle divisions if it were positive.

“S” on a graph

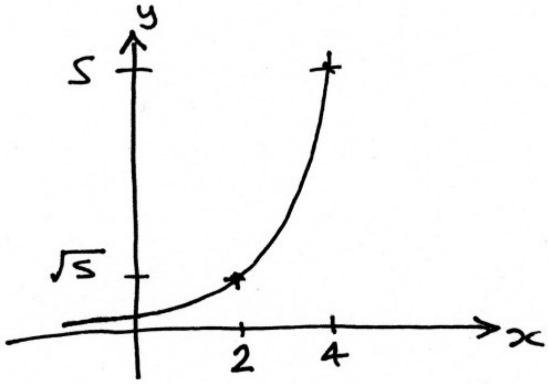
The connection between “S” and the circle on the Complex plane can be made slightly more intuitive by drawing graphs. We will take the base for a circle divided into 4 parts: 4.810477380965. This number to the power of just “x”, looks roughly like this on a graph:



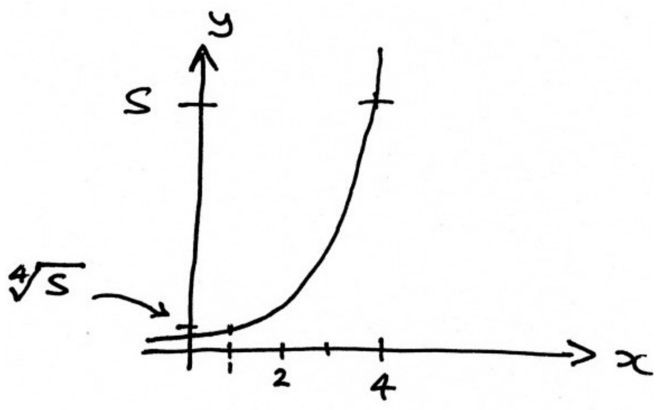
When the y-axis value is “S”, the x-axis value is 4, which is the number of parts into which the corresponding circle on the Complex plane has been divided.



When the x-axis value is 2, the y-axis value is $\sqrt{S}$, which is 23.140692632779 [to 12 decimal places].

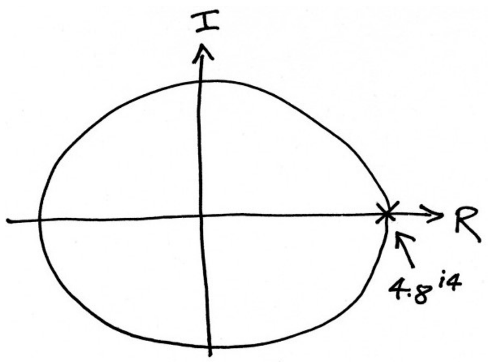


When the x-axis value is 1, the y-axis value is $\sqrt[4]{S}$, which is 4.810477380965 [to 12 decimal places].

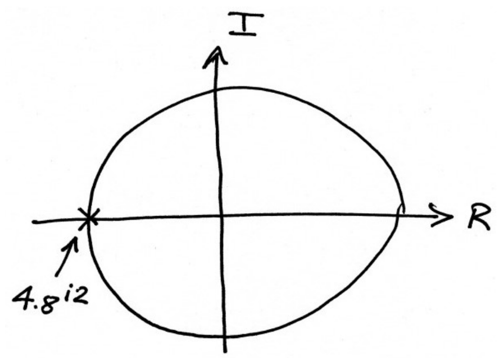


We will now look at points on the circumference of a unit-radius circle on the Complex plane, the positions of which are indicated using "$4.810477380965^{i\theta}$".

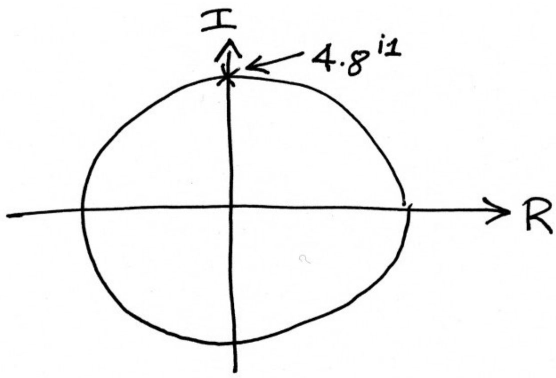
When x is 4 (when we have 4.810477380965^{i4}) , the point is all of the way around the circle:



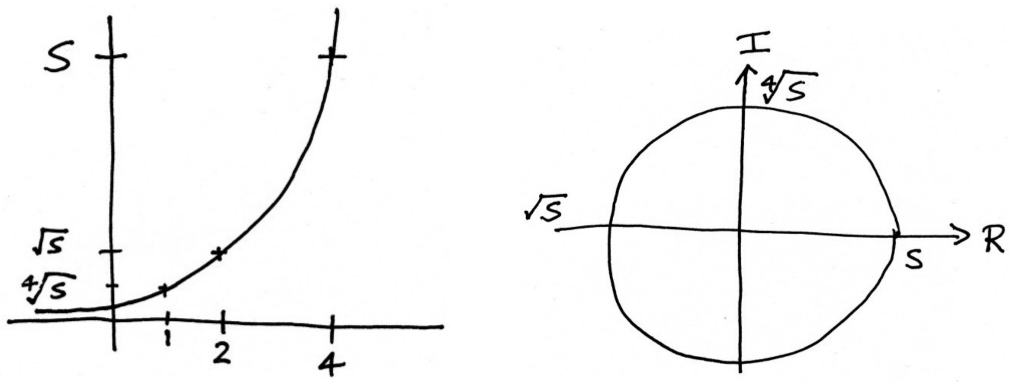
When x is 2 (at the square root of 4.810477380965^{i4}), the point is half way around the circle:



When x is 1 (at the fourth root of 4.810477380965^{i4}), the point is quarter of the way around the circle:



It is as if the y-axis values on the graph for 4.810477380965^{x} are being wrapped around the circle, but in a logarithmic way.

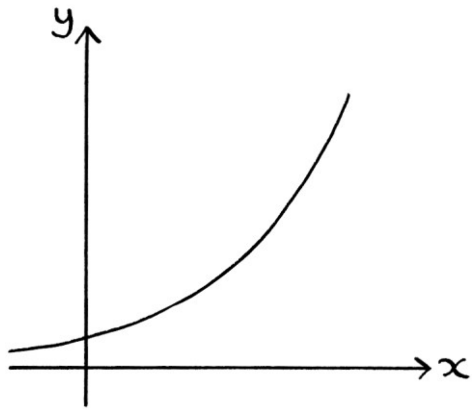


The same is true for any other base and exponent combination that obeys the formula: c = $\sqrt[x]{S}$.

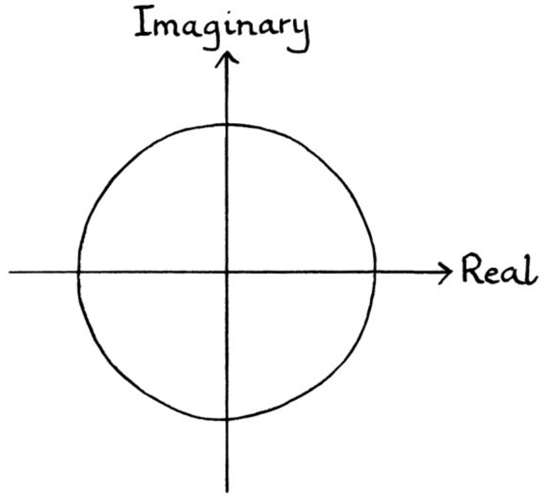
Chapter 7: Increasing powers of "i"

[This chapter has better pictures than the other chapters, as they are taken from my book about waves, available at www.timwarriner.com]

If we have a Real number to a power of "x", it will indicate a series of points on an exponential curve on x and y-axes:



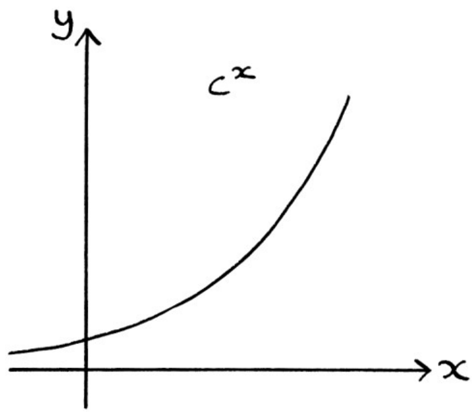
... and if we have a Real number to a power of "ix" (or in other words, "iθ"), it will indicate a series of points on a unit-radius circle on the Complex plane:



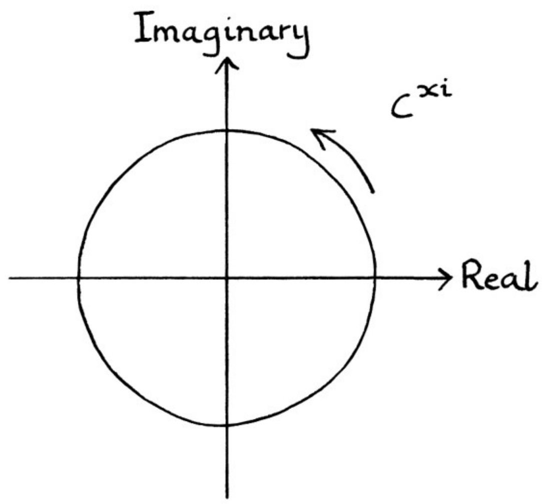
The 4-stage cycle of "i" powers

Where these ideas become more interesting is when we repeatedly raise an existing exponential to the power of "i". As an example, we will use a generic formula: "c^x" where "c" is any Real number over 1.

If we were to plot the points "$y = c^x$" over a range of "x", we would end up with a curve on x and y-axes as in this picture. As "x" increases, so does the result of "c^x".



If we now raise this entire exponential to the power of "i", we will get: $(c^x)^i = c^{xi}$. In this formula, "x" is actually an angle in a system of angles related to the base "c". If we were to plot the points of "c^{xi}" for a range of "x", we would get a unit-radius circle. As "x" increases, "c^{xi}" indicates a point on a unit-radius circle at ever-increasing angles, which is the same thing as saying that as "x" increases, "c^{xi}" indicates points that are further anticlockwise around the circle:



Now, if we raise the entire exponential "c^{xi}" to the power of "i", we will have:

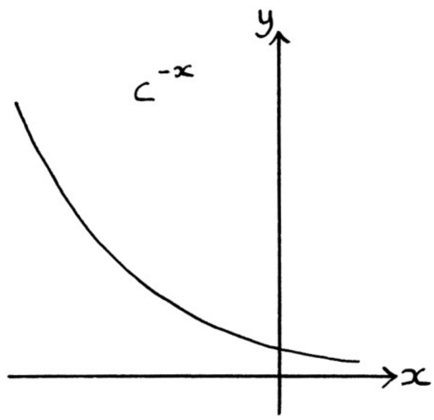
$(c^{xi})^i$

$= c^{xii}$

$= c^{(x * -1)}$

$= c^{-x}$

We have gone from a circle on the Complex plane, back to a curve on x and y-axes. What is more, this time the curve is "c^{-x}" which is a mirrored version of the exponential curve that we started with. As "x" increases, the result of "c^{-x}" decreases. The curve looks like this:

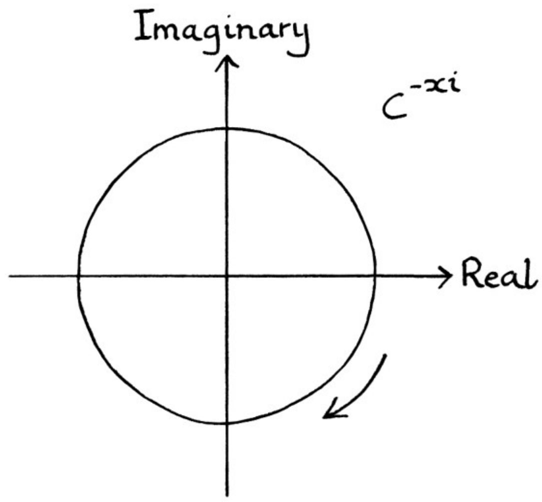


Now, if we raise the entire exponential, "c^{-x}" to the power of "i", we will have:

$(c^{-x})^i$

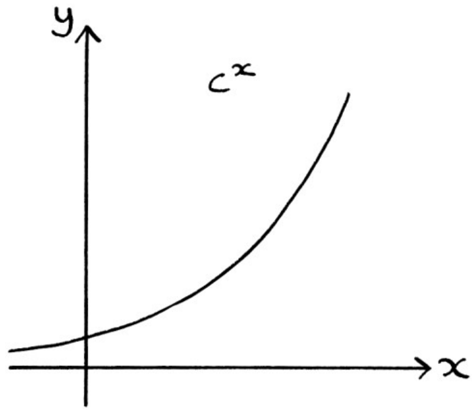
$= c^{-xi}$

We have gone from the backwards exponential curve to a circle on the Complex plane again. However, this time, as "x" increases, "c^{-xi}" indicates a point on a unit-radius circle at ever *decreasing* angles. In other words, as "x" increases, "c^{-xi}" indicates points that are further *clockwise* around the circle.

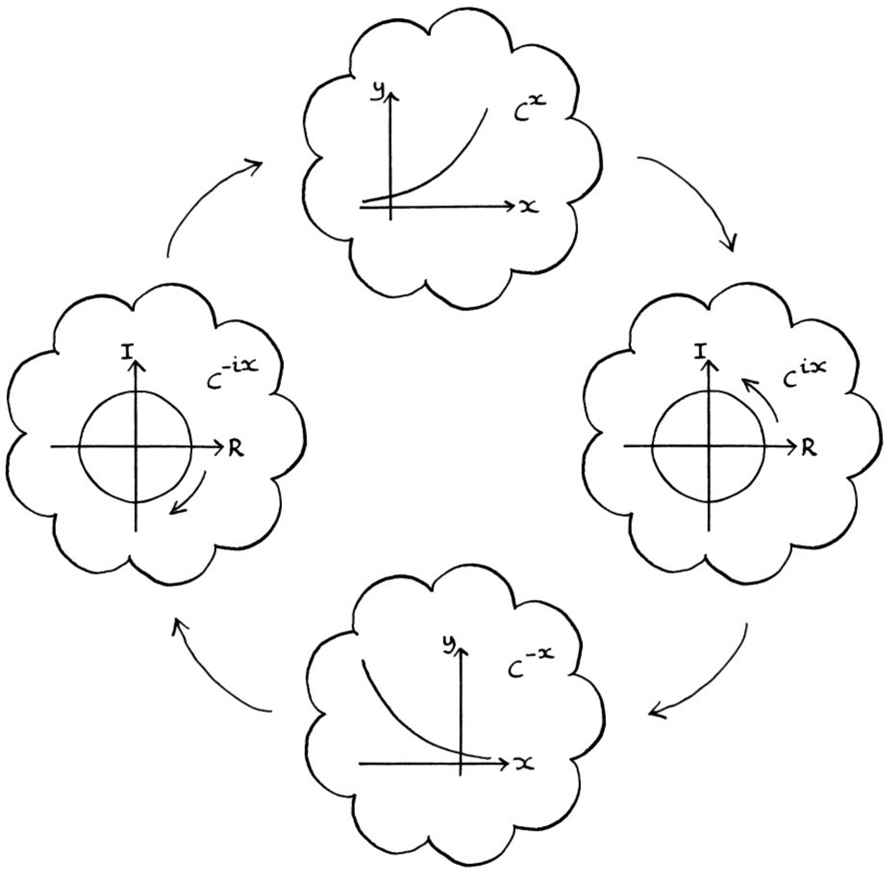


And, if we raise "c^{-xi}" to the power of "i", we get:
$(c^{-xi})^i = c^{-xii} = c^{(-x * -1)} = c^x$

We have ended up with the curve on x and y-axes that we started with:



To summarise this series of events, every time we raise something to the power of "i" we switch from a curve on x and y-axes to a circle on the Complex plane, or we switch from a circle on the Complex plane to a curve on x and y-axes. There is also a progression, or cycle, of four stages that goes: curve, circle, backwards curve, backwards circle. At whichever stage we start, raising the exponential to the power of "i" will move it onto the next stage:



An example with "e"

To show the cycle working in practice, we will look at what happens to "e^x". [Note that this is "e^x" and not "e^{ix}"]

"e^x" is a curve on x and y-axes. When raised to the power of "i", it becomes:
$(e^x)^i = e^{xi} = e^{ix}$
... which is a circle on the Complex plane.

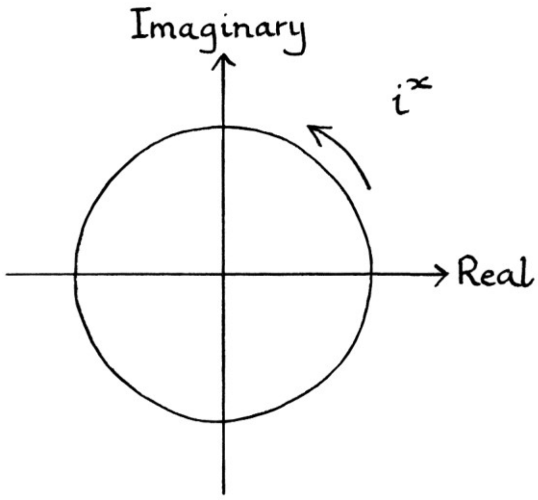
When that is raised to the power of "i", we get:
$(e^{ix})^i = e^{xii} = e^{x * -1} = e^{-x}$
... which is a "backwards" curve on x and y-axes.

When that is raised to the power of "i", we get:
$(e^{-x})^i = e^{-xi} = e^{-ix}$, which is a "backwards" circle on the Complex plane. The higher "x" is, the further *clockwise* the result will be around the circle.

When that is raised to the power of "i", we get:
$(e^{-ix})^i = e^{-ixi} = e^{-xii} = e^{(-x * -1)} = e^x$
... which is what we started with.

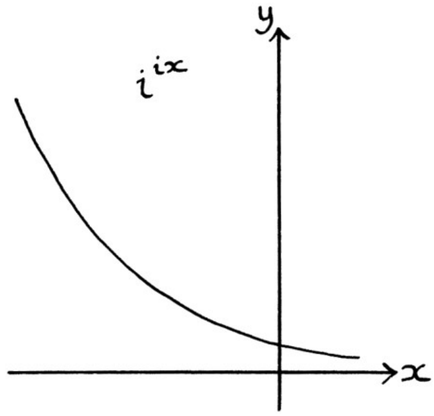
An example with "i"

The formula "i^x" is already referring to a circle on the Complex plane. Therefore, it is at stage 2 in the cycle. As "x" in "i^x" increases, the further anticlockwise around the circle the result will be.



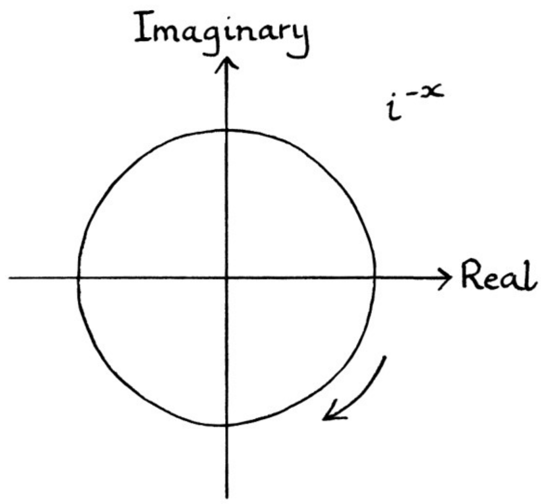
When "i^x" is raised to the power of "i", we get:
$(i^x)^i$
$= i^{ix}$
... which refers to a backwards curve on x and y-axes.

This means that for any value of "x", the result of "i^{ix}" will always be a Real number. The higher "x" is, the smaller the result will be.

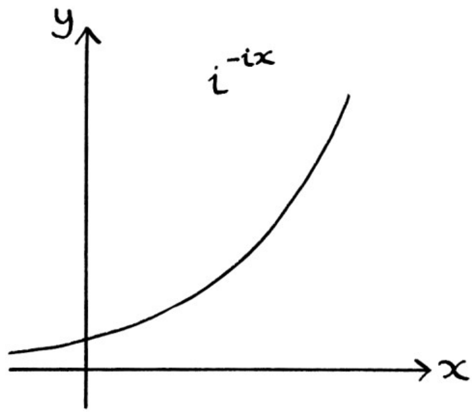


When that is raised to the power of "i", we get:
$(i^{ix})^i$
$= i^{iix}$
$= i^{-x}$
... which refers to a backwards circle on the Complex plane.

This means that for any value of "x", the result of "i^{-x}" will be a Complex number. The higher "x" is, the further *clockwise* around the circle the result will be.



When "i^{-x}" is raised to a power of "i", we get:
$(i^{-x})^{i}$
$= i^{-ix}$
... which refers to a forwards curve on x and y-axes. The higher "x" is, the higher the result will be.



When "i^{-x}" is raised to a power of "i", we get:
$(i^{-ix})^{i}$
$= i^{-iix}$
$= i^{x}$
... which is what we started with, and refers to a forwards circle on the Complex plane.

"i" to the power of "i"

Where the four-stage cycle becomes useful is in visualising the solving of seemingly difficult calculations such as "i^{i}" or " $\sqrt[i]{i}$ ". There are already methods to calculate such things using the fact that "$e^{i\theta} = \cos\theta + i\sin\theta$" [in radians], but they take some thought. We can visualise what we are doing more easily using the knowledge of the four-stage cycle.

We know that "i^{x}" represents a circle on the Complex plane. Therefore, if we raised that to the power of "i", we know that the resulting formula, "i^{ix}", would be at the next stage in the four-stage cycle. Therefore, it would be a backwards curve on x and y-axes. In that case, any results would always be Real numbers. In other words, for any Real value of "x", "i^{ix}" would be a Real number. Therefore, if "x" is 1, we would have "i^{1i}", which is "i^{i}", and we would know that the solution must be a Real number.

Actually calculating the value of "i^i" is easy. We know that "i^x" divides a circle up into 4 pieces. Using what we have learnt from earlier in this explanation, this means that it is identical to "4.8105^{ix}". Therefore, "i^{ix}" is identical to "4.8105^{iix}", which ends up as "4.8105^{-x}". This confirms that "i^{ix}" is a backwards exponential curve on x and y-axes. The value "i^i" is equal to "i^{1i}", which means it is the same as 4.8105^{-1}, which is 1 ÷ 4.8105 = 0.20787958 [to 8 decimal places, and calculated from the full value of 4.810477380965...]

Usually in explanations of calculating "i^i", people are often surprised that "i^i" should result in a Real number. This is because they do not realise that raising something to the power of "i" changes the nature of what the exponential is about. Previously, one might have tried to plot the result of "i^i" on the Complex plane – in other words, by marking its position at "0.2079 + 0i", and in doing that, it seems very confusing that something should end up rotated and scaled to that point on the Real axis. However, as we now know, the value 0.2079 in this case does not belong on the Complex plane – it belongs on the backwards exponential curve on x and y-axes. It represents the point on the curve of "4.8105^{-x}" where x is 1.

i^{th} roots of i

An i^{th} root of "i" such as " $\sqrt[i]{i}$ " seems completely incomprehensible, but it is actually reasonably easy to solve. [We actually discovered the value of " $\sqrt[i]{i}$ " when we found the base for a circle with 4 divisions earlier in this chapter, but we did not really *solve* it – instead we just knew what its equivalence was].

We will solve a generic root: the ix^{th} root of i: $\sqrt[ix]{i}$. Given how exponentials and roots work, this can be rephrased as:

"$i^{(1 \div ix)}$"

... which we will write as:

"$i^{(1 / ix)}$"

This is now "i" raised to what is, essentially, a multiple of "i". From our knowledge of the four-stage cycle, we know that "i" raised to a multiple of "i" will have a result that is on the backwards exponential curve on the x and y-axes. Therefore, the result will be a Real number.

We know that "i^x" is equivalent to "4.8105^{ix}". Therefore, "$i^{(1/ ix)}$" will be equivalent to:

$4.8105^{(i * 1/ix)}$

$= 4.8105^{(i/ix)}$

$= 4.8105^{(1/x)}$.

Therefore:

$\sqrt[i]{i}$

= $\sqrt[1i]{i}$

= $i^{(1/1i)}$

= $4.8105^{(1/1)}$

= 4.8105

As another example, if we wanted to calculate:

$\sqrt[3i]{i}$

... it would be equivalent to:

$i^{(1/3i)}$

= $4.8105^{(1/3)}$

= 1.68809179

[to 8 decimal places, and calculated from the full value of 4.810477380965...]

Chapter 8: Some thoughts

The problem with “S”

There is one big problem in this explanation: we only have one number that we can use to test the validity of “S”. If we did not have a way of calculating “e” raised to a power of “i”, we would not be able to calculate “S”. We are calculating “S” using “$e^{2\pi}$”, and everything else is derived from that. The values of other bases, what “i^i” is, what the i^{th} root of a number is, and so on, all depend on how we calculate “S”. If we had said that “S” were a different number, everything else would have a consistent manner of calculation, but the results would be different. However, if we did not know about “$e^{2\pi}$”, we would not be able to say that the results were wrong because, as far as I can tell, there is no other way of knowing that. The values of everything after we rebased “i^θ” with a Real base rely solely on “$e^{2\pi}$”. Ideally, we would have other ways of calculating “S” that do not require knowledge of “e”.

The meaning of $e^{i\theta}$

As we know, using “i^θ” to identify the position of a point on a unit-radius circle’s edge is really a method of showing the amount of rotation required to get “1 + 0i” to that particular position. A multiplication by “i^θ”, where “θ” is an angle in a system that divides a circle into 4 parts, results in a point being rotated by that angle. The exponential “i^θ” can be used in two ways: it can be used for rotating a point around the Complex plane, or it can be used to identify the position of a point (by using its rotational properties).

If we, for example, multiply “i^2” by the number 3, we could say two things:

- We could say we are scaling a unit-radius circle by 3, and so the point on the unit-radius circle at the equivalent of 180 degrees is now at the same angle, but 3 times further away from the origin.
- We could say that we are rotating the point “3 + 0i” by 180 degrees by multiplying it by “i^2”.

These both amount to the same thing. The result is a point at “−3 + 0i”. The first way of thinking is possibly the easiest to visualise, but the second is, in my view, a better way of understanding the properties of “i^θ”. It reinforces the rotational aspect of a multiplication by “i^θ”.

The rotation idea makes things more intuitive when the maths becomes more complicated. For example, if we multiply “3.4 − 7.8i” by “i^3”, we know that we are really rotating that point by the equivalent of 270 degrees. The result will be exactly the same distance from the origin.

Some deductions

We know that multiplying a Complex number by “i^{θ}” results in a rotation of the point indicated by that number by “θ” quarter-circle angle units.

We know that the formula of “i^{θ}” can be rephrased so that the “θ” can be used to refer to angles in other angle systems. Each resulting formula says an identical thing, but we adjust the size of “θ” by scaling it by a value, so that we can use other angle systems.

Therefore, multiplying the position of a point by one of these rephrased formulas will still result in a rotation of the point by the angle of “θ”, where “θ” is an angle in the relevant system for that formula.

We know that “$i^{(2\theta/\pi)}$” is a rephrasing of “i^{θ}”, where the “θ” in “$i^{(2\theta/\pi)}$” is an angle in radians. Therefore, multiplying a point by “$i^{(2\theta/\pi)}$” will result in the point being rotated by “θ” radians.

We know that “$(\sqrt[0.5i\pi]{i})^{i\theta}$” is a rephrasing of “$i^{(2\theta/\pi)}$”, but with (what is ultimately) a Real base to an Imaginary power, instead of an Imaginary base to a Real power. Therefore, multiplying a point by *that* will result in the point being rotated by “θ” radians.

We know that “$e^{i\theta}$” is a rephrasing of $(\sqrt[0.5i\pi]{i})^{i\theta}$, but with an obviously Real base, instead of a base that has an i^{th} root of “i” in it.

From all of that, we can say that multiplying a Complex number by “$e^{i\theta}$” rotates the point indicated by that number by “θ” radians.

And from that, we can conclude that multiplying any number by “$e^{i\theta}$” is equivalent to treating that number as if it were a Complex number identifying a point, and then rotating it by “θ” radians.

Depending on how much you already knew, such deductions might be obvious, or they might be slightly profound. However, the basic idea from all this is as follows: we used to treat "i^{θ}" as merely something that *rotates* a point, but now we can use it to *identify* points. Conversely, we used to treat "$e^{i\theta}$" as something that *identifies* a point, but now we know that we can use it to *rotate* a point. It just so happens that anything that rotates a point can be used to identify a point too. The main idea here is that "$e^{i\theta}$" is in essence a means of rotation, and we use that rotation to identify the position of points.

Because we are so used to seeing "$e^{i\theta}$" as a means of identification, it might be difficult to appreciate that it is no different to "i^{θ}" in how it can be used for rotation.

Now we know that if we multiply a Complex number such as "5.5 – 7.8i" by "e^{2i}", we are really rotating it by 2 radians. The result will be the same distance from the origin.

Similarly, if we multiply a number such as 10 by "e^{3i}", we are really multiplying the number "10 + 0i" by "e^{3i}", which is the same as rotating the point described by "10 + 0i" by 3 radians. The result, again, will be the same distance from the origin (which in this case is 10 units).

If we multiplied the number 2 by "e^{1i}", 2π times in a row, we would end up with the number 2. Each step rotates it one more radian onwards. [We can also know this is true because it would be "$2 * (e^{i})^{2\pi}$", which is "$2 * e^{2\pi i}$" = 2 * 1 = 2].

What "$e^{i\theta}$" means

After all that, we can say that multiplication by "$e^{i\theta}$" is a method of rotating a point on the Complex plane by "θ" radians at a time. It should be thought of as being the exact same concept as "i^{θ}" (but obviously rotating by a different amount). And, as with "i^{θ}", "$e^{i\theta}$" can also be used to identify the position of a point on a unit-radius circle on the Complex plane, by showing how much rotation is required to get "1 + 0i" to that particular point.

We can also say something similar about all of the exponentials in this explanation. Every exponential is ultimately a means of rotating a point by an angle of "θ", where "θ" is an angle in that particular system. The exponentials can also be used to *indicate* the position of a point by using their rotating properties.

Another idea is that if we really wanted to emphasise the rotational nature of “$e^{i\theta}$”, when we say:
“$e^{i\theta} = \cos\theta + i \sin\theta$”
... we could instead say:
“$(1 + 0i) * e^{i\theta} = \cos\theta + i \sin\theta$”
... which, in English, means:
“1 + 0i”, when multiplied by “$e^{i\theta}$” results in the point at “$\cos\theta + i \sin\theta$”, when Cosine and Sine are working in radians
... which ultimately means the same thing as:
“1 + 0i”, when rotated by “θ” radians, results in the point at “$\cos\theta + i \sin\theta$”.

If we look at the equation:
“$e^{i\pi} = -1$”
... we could write it out in full as:
“$(1 + 0i) * e^{i\pi} = -1 + 0i$”
... or, in English, as:
“1 + 0i”, when multiplied by “$e^{i\pi}$”, results in the point at “−1 + 0i”.
... which ultimately means the same thing as:
“1 + 0i”, when rotated by π radians, ends up at “−1 + 0i”

Simpler maths

The rotation idea of “$e^{i\theta}$” and other related exponentials makes visualising some calculations a lot easier.

If we want to rotate the point “2.2 + 4.7i” by say 3 radians, we can now understand how this would work:
$(2.2 + 4.7i) * e^{i3}$

If we want to rotate the point “1.87 – 9.76i” by 22.5 *degrees*, we could use:
$(1.87 - 9.76i) * 1.017606491206^{22.5i}$

If we want to rotate the point “6 + 6i” by 0.23 whole-circle angle units, then we could use:
$(6 + 6i) * 535.491655524765^{0.23i}$

Solving S in a different way

Knowing that “$e^{i\theta}$” and all the other exponentials are primarily a means of rotation, suggests to me that we could use that idea to calculate bases without using knowledge of “$e^{2\pi}$”, although, I have not yet thought of a way to do this.

Chapter 9: Waves

If we put every possible value of “θ” quarter-circle angle units into the formula “i^{θ}”, we will draw a circle. Similarly, if we put every possible value of “θ” radians into the formula “$e^{i\theta}$”, we will also draw out a circle. We can incorporate time into both formulas to indicate an object rotating around a circle. For example, “i^{4t}” will give the position of an object rotating around a unit-radius circle at one cycle per second at any time in seconds. The formula “$e^{2\pi it}$” will also give the position of an object rotating around a unit-radius circle at one cycle per second at any time in seconds. We can think of these formulas as representing objects rotating around circles, or we can think of them as representing objects rotating around helices. This is because the path around a circle as portrayed over time on a three-dimensional chart with time as the third axis will be a helix. Therefore, over time, “i^{4t}” and “$e^{2\pi it}$” both draw out circles or helices, depending on how we choose to view them.

Depending on your knowledge of maths, you might know that we can portray time-based waves in terms of Imaginary exponents of “e”. When we do this, we are actually creating two-dimensional waves in a three-dimensional Complex plane by adding or subtracting helices with equal but opposing frequencies.

In this way, the general Cosine wave formula consists of half the sum of two helices (represented by Imaginary powers of “e”) with opposing frequencies. It is as so:

$$\frac{1}{2} * (ae^{i(2\pi ft\,+\,\phi)} + ae^{-i(2\pi ft\,+\,\phi)})$$

... where:

- “a” is the amplitude of the wave
- “f” is the frequency of the wave
- “t” is the time in seconds
- “ϕ” is the phase in radians

This formula is mathematically identical to “a cos ((2π * ft) + ϕ)”.

The general Sine wave formula consists of the subtraction of one helix from another (both represented by Imaginary powers of "e") with opposing frequencies, then halved and rotated clockwise. [The division by "i" rotates everything clockwise.] It is as so:

$$\frac{1}{2i} * (ae^{i(2\pi ft\, +\, \phi)} - ae^{-i(2\pi ft\, +\, \phi)})$$

This formula is mathematically identical to "a sin ((2π * ft) + ϕ)".

We can also portray waves in a similar way using powers of "i". We are still adding or subtracting helices with opposing frequencies. The only difference is that we are portraying the helices with powers of "i". The general formula for a Cosine wave is as so:

$$\frac{1}{2} * (ai^{(4ft\, +\, \phi)} + ai^{(-4ft\, -\, \phi)})$$

... where:

- "a" is the amplitude of the wave
- "f" is the frequency of the wave
- "t" is the time in seconds
- "ϕ" is the phase in *quarter-circle angle units*.

The general formula for a Sine wave is as so:

$$\frac{1}{2i} * (ai^{(4ft\, +\, \phi)} - ai^{(-4ft\, -\, \phi)})$$

Chapter 10: Conclusion

What have we learnt?

From everything in this explanation, we now know:

- We can identify a point around a unit-radius circle using just a power of “i”.

- If we do this, the exponent acts as an angle in quarter-circle angle units.

- If we do this, we can also say that that point is at “cos θ + i sin θ”, where “θ” is a quarter-circle angle unit, and Cosine and Sine are operating within this same system.

- We can scale this exponential to identify a point anywhere on the Complex plane. Either we can consider ourselves scaling the exponential by a particular value, or we can consider ourselves rotating that value by multiplying it against the exponential.

- We can adjust this method so that “θ” can be a value representing an angle in a different system of dividing up a circle. For example, if we want “θ” to be in degrees, we can use “$i^{\theta/90}$”. Any point referred to in this way can also be referred to using “cos θ + i sin θ”, where “θ” is an angle in degrees, and Cosine and Sine are working in degrees (but only degrees). If we want “θ” to be in radians, we can use “$i^{2\theta/\pi}$”. Any point referred to in this way can also be referred to using “cos θ + i sin θ”, where “θ” is an angle in radians, and Cosine and Sine are working in radians (but only radians).

- We can rephrase the “i^{θ}” exponential and the various exponentials that came from it, in the form of a base raised to an exponent that is a multiple of “i”. These bases will mention an i^{th} root of “i” in some form.

- Using the knowledge from any known method of raising a value to an Imaginary power for which we know the result, we can convert the untidy bases from the rephrased exponentials into Real numbers. [I only know two methods of raising a value to an Imaginary power that have a known result – the Taylor series for “e” and the idea of compound interest – and both of those involve raising “e” to the power of “i”. There must be others though].

- Given that all of these exponentials are ultimately derived from “i^θ”, and it is the case that “$i^\theta = \cos\theta + i \sin\theta$”, when “θ” is quarter-circle angle units, and Cosine and Sine are operating in that system, then this relationship still holds – all of the bases when raised to the power of “iθ” will be equal to “$\cos\theta + i \sin\theta$”, when “θ” is an angle in that particular system of dividing up a circle, and Cosine and Sine are operating in the relevant system.

- All the exponentials, including “$e^{i\theta}$”, are primarily forms of rotation, however, in the case of “$e^{i\theta}$”, we tend not to think of it in this way. An exponential written as “$c^{i\theta}$” could also be thought of as being “1 + 0i” multiplied by “$c^{i\theta}$”.

- A multiplication by an exponential, “$c^{i\theta}$”, rotates a point on the Complex plane by the angle “θ”, where “θ” is an angle in a system of angles implied by the base.

- A multiplication by “$e^{i\theta}$” rotates a point on the Complex plane by “θ” radians.

- A multiplication by “e^{i}” rotates a point on the Complex plane by 1 radian.

- Repeatedly raising an exponential to the power of “i” changes the realm to which the exponential applies, by switching through a 4-stage cycle. The stages are: curve, circle, backwards curve, backwards circle. Knowing this allows us to know whether a result of an exponential will be a Real number or not.

Thoughts on “$e^{i\theta}$”

I will comment on some of the properties of “$e^{i\theta}$” that seem to impress or confuse people the most, now that we know what we know.

“$e^{i\theta}$” identifies the position of a point on a unit-radius circle

We now know that the same is true for countless other exponentials.

We can use “$e^{i\theta}$” to describe waves

We can also use countless other exponentials to describe waves too.

"$e^{i\theta}$" shows that there is a connection between "e" and "π"

This is still a very interesting fact. There might be other interesting numbers that are also connected.

"$e^{i\theta}$" gives validity to radians

If you are used to using radians, and then one day you find out that "$e^{i\theta}$" works in radians, it is mind-blowing. It makes the idea of dividing a circle up into 2π divisions seem like the natural way of dividing up a circle. If we look *solely* at exponentials, it should be apparent that it is difficult to say that one particular way of dividing up a circle is any more natural than another. However, the breakthrough into circles on the Complex plane usually comes from solving "e^i" and not through solving other exponentials, which is something that gives radians more validity than other angle systems. Dividing a circle into 2π divisions is definitely the simplest way for Imaginary exponents. There are reasons to use radians in maths other than the behaviour of "$e^{i\theta}$" – for example, radians are by far the most useful angle system in calculus.

If we were choosing angle systems for the purposes of exponentials, I would say that there are two "natural" ways of dividing up a circle – with 4 divisions and with 2π divisions.

"$e^{i\theta} = \cos\theta + i\sin\theta$" when dealing in radians

After reading this explanation, this equivalence should stop being surprising. If someone now says, "$e^{i\theta}$ equals $\cos\theta + i\sin\theta$", the response should be, "Well, yes. It would do." If something identifies a point on a unit-radius circle at a particular angle, then it will always be the case that that same point can be described using the Cosine and Sine of that angle, when Cosine and Sine are working in the same angle system.

Cosine and Sine are *defined* as the functions that find the x-axis and y-axis coordinates of a point at a particular angle on the circumference of a unit-radius circle. Therefore, any other way of doing the same thing will have to be equal to "$\cos\theta + i\sin\theta$" on the Complex plane, when Cosine and Sine are working in the same angle system.

“$e^{i\theta}$” is not equal to “cos θ + i sin θ” when not dealing in radians

You will often see examples in books and on the internet where people *mistakenly* think that “$e^{i\theta}$” can work just as well in degrees as in radians. An Imaginary exponent of “e” will always indicate a point on a unit-radius circle at an angle of the non-Imaginary part of the exponent in *radians*. The equation “$e^{i\theta} = \cos\theta + i \sin\theta$” is not true if Cosine and Sine are working in degrees (or any angle system other than radians). This should be obvious because any point on a unit-radius circle at an angle of a particular number of radians will not be in the same place as a point at that same number of degrees.

The most common example of this mistake is the incorrect time-based formula “e^{360it}”. [The multiplication by 360 is intended to be the degree version of the “2π” frequency correction that gives a default frequency of 1 cycle per second.] The presence of 360 implies that the writer of the formula is intending to use degrees. For any particular value of “t”, “e^{360it}” is not equal to “cos 360t + i sin 360t” when Cosine and Sine are working in degrees. If the path of an object rotating around a circle follows the formula“e^{360it}”, then the object’s frequency is not, as some people might think, 1 cycle per second, but instead 57.2958 cycles per second [57.2958 is 360 ÷ 2π]. If someone insists on using “e^{360it}”, then the equivalence is actually:
“$e^{360it} = \cos (360 * 57.2958t) + i \sin (360 * 57.2958t)$
... when Cosine and Sine are working in *degrees*.

Although “e^{360it}” on its own is mathematically valid, it would be extremely rare that its actual meaning was what was intended.

I think some people make the mistake of believing that an Imaginary exponent of “e” can work in degrees because they never truly understood the basics of waves. I hope that, after having read this explanation, you can see why they are wrong.

"$e^{i\pi} = -1$"

I would say that there are four different responses that people have to this equation, depending on their level of understanding:

- They do not understand what it means. No one can be blamed for not understanding the formula as it is only relevant in particular fields.

- They do not *really* understand what it means, and are amazed by this formula.

- They understand what it means, but treat its significance in being *what* it says, so are not that amazed by it. In other words, they see "$e^{i\pi} = -1$" as being "$e^{i\pi} = -1 + 0i$", which just states that a point at the equivalent of 180 degrees on a unit-radius circle is situated at the coordinates (−1, 0). Given that "$e^{i\theta}$" is "$\cos\theta + i\sin\theta$" when Cosine and Sine are working in radians, the position of that point is to be expected, so it does not really tell us anything we did not already know.

- They understand what it means, and treat its significance in being the fact that it is possible to say what it says *in this way*, and therefore they *do* find it interesting. In this sense, the interesting part of "$e^{i\pi} = -1$" is the fact that without how exponentials and circles on the Complex plane relate to each other, the formula would not make any sense. It *is* interesting how "$e^{i\theta}$", or any base raised to "$i\theta$", or even "i^{θ}", identifies a point at a particular angle on a unit-radius circle on the Complex plane. It is also very interesting how "e" and "π" are related. In the same way that the owner of the world's best fountain pen might write some text to show off the abilities of the pen, so is the formula "$e^{i\pi} = -1$" really a demonstration of the interesting underlying maths. In that sense, the formula is interesting, and is still interesting even after this entire explanation. It is still a good, succinct way of saying everything that is possible. It is difficult to come up with a better equation that summarises so much so succinctly.

Future Research

I think it would be interesting if anyone can come up with a method to calculate the Real number base for a system where a circle is divided into a certain number of portions, without resorting to knowing about "e^{i}". There must be other ways of bridging the gap from "i^{θ}" to "$c^{i\theta}$".

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